Math 21a Practice Hourly 2 Answers

1. a) The tangent plane is the plane through the given point which is orthogonal to the gradient. Here, $\nabla g = (-3y + z + 2, 2y - 3x, x - 1)$. At $(-1, 0, -1)$, one has $\nabla g = (1, 3, -2)$. Thus, the tangent plane consists of the points (x, y, z) which obey $x + 3y - 2z - 1 = 0$.

b) A normal vector to the plane where $x = y$ is the vector $(1, -1, 0)$. This is proportional to ∇g when $x = 1$ and $z = y + 1$. Furthermore, a point $(1, y, z = y + 1)$ has $g = 0$ only when $y^2 - 3y + 2 = 0$, which has $y = 1$ or $y = 2$. Thus, $P = (1, 1, 2)$ or $P = (1, 2, 3)$.

c) $L(x, y, z) = x + 3y - 2z - 1$ has the same value as g at $(-1, 0, -1)$ and the same gradient.

d) Any vector of **u** which obeys $\nabla g \cdot \mathbf{u} = 0$. Thus, $\mathbf{u} = (-3a + 2b, a, b)/(10a^2 + 5b^2 - 12ab)^{1/2}$ where both a and b are not zero. For instance, $\mathbf{u} = (-3, 1, 0)/(10)^{1/2}$.

2. a) The hottest point on the surface is $(\sqrt{3}/2, 0, 1/2)$.

b) The hottest point inside or on the surface is $(\sqrt{3}/10, 0, 10)$.

c) The coldest point is $(-\sqrt{3}/2, 0, -1/2)$.

The extreme points on the surface are obtained by solving the Lagrange multiplier equations for those points where $x^2 + y^2 + z^2 = 1$ and $\nabla T = \lambda \nabla g$ where $g(x, y, z) = x^2 + y^2 + z^2$. Here, $\nabla T = 10 (\sqrt{3}, 0, 1) - 100 (x, y, z)$ and $\nabla g = (2x, 2y, 2z)$. The extreme point inside is obtained by solving for the points where $x^2 + y^2 + z^2 < 1$ and $\nabla T = 0$.

3. a) $\mathbf{r}'(0) = \mathbf{i} + \mathbf{j}$ and $\mathbf{r}''(0) = 2\,\mathbf{j}$.

b) $\mathbf{s}'(0) = -\,\mathbf{j}$ and $\mathbf{s}''(0) = -\,\mathbf{i}$.

c) Write $\nabla T = a\,\mathbf{i} + b\,\mathbf{j}$. Then we are told that $\nabla T \cdot \mathbf{r}'(0) = a + b = 3$ and $\nabla T \cdot \mathbf{s}'(0) = -b = -1$. Thus, $b = 1$ and $a = 2$ and $\nabla T = 2\,\mathbf{i} + \mathbf{j}$ at $(1, 0)$.

4. a) The critical points occur where $\nabla f = (2xy - 4y, x^2 - 4x + y^2) = 0$. These are $(2, \pm 2)$, $(0, 0)$ and $(4, 0)$.

b) $(2, 2)$ is a local minimum, $(2, -2)$ is a local maximum, $(0, 0)$ is a saddle, $(4, 0)$ is a saddle. The 2nd derivative test establishes these assertions since the matrix f'' of 2nd derivatives has $\det(f'')$ and $\mathrm{trace}(f'') > 0$ at $(2, 2)$, and $\det(f'') > 0$ and $\mathrm{trace}(f'') < 0$ at $(2, -2)$ and $\det(f'') < 0$ at $(0, 0)$ and $(4, 0)$.

c) The direction of maximum increase is $(1, 1)/\sqrt{2}$. That of maximum decrase is $(-1, -1)/\sqrt{2}$.

d) Any vector of the form $(a, 0)$ with $a \neq 0$.

5. The best linear approximation to f at $(0, 25)$ is $L(x, y) = 5 + 5x + (10)^{-1} (y - 25)$. Using L to estimate f gives $5 + .5 + .03 = 5.53$.

6. After doing the y integration, one is left with integrating $2^{-1} x^3 (1 - x^2)$ between 0 and 1. The integral is 1/24.

7. Do the y integral first. (There is no closed form expression for the x integral if that one is done first.) The range for the y integral is from $y = 0$ to $y = x$. The resulting x integral is for the function $2x\ e^{x^2}$ with the range going from $x = 0$ to $x = 1$. Changing variables to $u = x^2$ shows that this is the same as the integral of e^u from $u = 0$ to $u = 1$, which is $e - 1$.