Math 21a Hourly 1 Answers

(Fall 1999)

Note: Problems 1-6 count 8 points each, while Problem 7 counts for 2 points.

1. Let $\mathbf{v} = (2, 1, 4)$ and $\mathbf{w} = \frac{1}{3}(1, 2, 2)$.
   a) Find the lengths of $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v} - 3\mathbf{w}$.
   b) Find the scalar projection of $\mathbf{v}$ in the direction of $\mathbf{w}$.
   c) Find $\mathbf{v} \times \mathbf{w}$.
   d) Find numerical values for the constants b and c so that both $\mathbf{v}$ and $\mathbf{w}$ are tangent to the plane where $bx + cy + z = 0$.

- Answer:
   a) $|\mathbf{v}| = \sqrt{21}$, $|\mathbf{w}| = 1$, $|\mathbf{v} - 3\mathbf{w}| = \sqrt{6}$.
   b) This number is $\mathbf{v} \bullet \mathbf{w} = 4$.
   c) $(-2, 0, 1)$.
   d) $b = -2$ and $c = 0$.

2. Let Π denote the plane where $2x - 2y + z = 3$.
   a) Find three points in Π which do not all lie on the same line.
   b) Find a non-zero vector which is perpendicular to Π.
   c) Find the distance from Π to the origin.
   d) Write an equation for a line which lies entirely in Π.

- Answer:
   a) $(3/2, 0, 0)$, $(0, -3/2, 0)$ and $(0, 0, 3)$. (There are infinitely many other possibilities)
   b) $(2, -2, 1)$, or any non-zero multiple
   c) The distance is 1.
   d) Parametric: Send $t \to (3t, 3t, 3)$. Nonparametric: $x = y$ & $z = 3 - 2x + 2y$. (There are infinitely many other possibilities.)

3. A particle moving in space has position at time t given by $(3 \sin(t^2), 3 \cos(t^2), 4\, t^2)$.
   a) Find the coordinates of the particle at $t = \sqrt{\pi}$.
   b) Find the velocity vector of the particle at $t = \sqrt{\pi}$.
   c) Find a parametric equation for the line tangent to the trajectory at $t = \sqrt{\pi}$.
   d) Find the distance traveled by the particle between $t = 0$ and $t = \sqrt{\pi}$.

- Answer:
  a) (0, -3, 4 π).
  b) 2 $\sqrt{\pi}$ (-3, 0, 4)
  c) t $\to$ (-3t, -3, 4 (π + t)).
  d) The velocity vector at general t is 2t (3 cos(t^2), -3 sin(t^2), 4) whose length is 10 t. Thus, the distance is the integral of this last function from 0 to $\sqrt{\pi}$ which is 5 $\sqrt{\pi}$.

4. Let **v** = (5, 3, 1) and **p** = (1, 0, 1). It turns out that the end points of the vectors **r** which obey (**r** - **p**) $\times$ **v** = 0 all lie on the same line, L.
   a) Find a point on L and a tangent vector to L.
   b) Write a parametric equation for L.
   c) Find a point on L with distance 5 from the origin.
   d) Find the closest distance from L to the origin.

- Answer:
  a) **p** is on L and **v** is tangent to L.
  b) t $\to$ **p** + t **v**.
  c) (-4, -3, 0); the case of t = -1 in the preceding parameterization.
  d) $d = |\mathbf{p} \times \mathbf{v}|/|\mathbf{v}| = \sqrt{\frac{34}{35}}$.

5. Let **v** = (5, 3, 1) and let L denote the line through the origin with tangent vector **v**.
   a) Find an equation for some plane through the origin which contains L.
   b) Let **w** = (-10, b, 2). Find a value for b which makes **w** perpendicular to **v**.
   c) Let **r** = (-10, b, c). Find values for b and c which makes **v** and **r** parallel.
   d) Suppose that **u** = (1, 3, -2) and **s** = (-30, -18, c). Find a value for c which makes **u** $\times$ **s** perpendicular to **v**.

- Answer:
  a) The plane where x - 3z = 0. (There are infinitely many other possibilities.)
  b) **w**•**v** = 0 requires b = 16.
  c) b = -6 and c = -2.
  d) If **s** is parallel to **v**, then **u** $\times$ **s** will be perpendicular to **v**. For this, take c = -6.

6. The position of a particle in space at time t is the head of a vector **r**(t) based at the origin. Use **k** to denote the vector (0, 0, 1) and suppose that **r**(t) - **k** (**k**•**r**(t)) is orthogonal at all times to the

vector $\mathbf{r}'(t)$ - $\mathbf{k}$ $(\mathbf{k}\bullet\mathbf{r}'(t))$. Also, suppose that the particle does not move along any one fixed line in space. Lable each of the statements below with

- 'A' if it is true for any and all $\mathbf{r}(t)$ as described above;
- 'B' if it is true for some, but not every $\mathbf{r}(t)$ as described above;
- 'C' if it is never true for any $\mathbf{r}(t)$ as described above.

a) The particle moves on the surface of a sphere centered at the origin.
b) The particle moves on the surface of a plane.
c) The particle moves on the surface of a cylinder.
d) The y coordinate of the particle is constant.

Please explain your reasoning in a sentence of two.

- Answer:
  a) B, for you can take $\mathbf{r}(t) = (\cos(t), \sin(t), 0)$. This lies on the surface of the radius 1 sphere and satisfies the conditions. But, $\mathbf{r}(t) = (\cos(t), \sin(t), t)$ also satisfies the condition, but traces out a corkscrew path.
  b) B, take the same $\mathbf{r}(t)$'s as in the answer for 6a for justification.
  c) A, indeed, write $\mathbf{r}(t) = (x(t), y(t), z(t))$ and then $\mathbf{r}$ - $\mathbf{k}$ $(\mathbf{k}\bullet\mathbf{r}) = (x(t), y(t), 0)$. Likewise, $\mathbf{r}'$ - $\mathbf{k}$ $(\mathbf{k}\bullet\mathbf{r}) = (x', y', 0)$, so our condition says that the motion in the x-y plane is just along a circle of fixed radius. Add in the z-coordinate (about which we know nothing) and we see that the motion is along a cylinder stretching off along the z-axis.
  d) C, if the motion in the x-y plane is along a circle, and the particle does not have constant x-y coordinate (which would mean its total motion was along a line parallel to the z-axis), then the y-coordinate must change with time since the y coordinate is not constant on a circle.

7. Let $\mathbf{u} = (3927, 42, -999)$ and $\mathbf{v} = (195, 735, 1115)$. What is $(\mathbf{u}/3 - \mathbf{v}/5)\bullet(\mathbf{u} \times \mathbf{v})$?

- Answer: 0. The vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.