Math 21a (Fall 99) Review Problems

- The problems given below sample the material from the course which you will be responsible for on the final examination.
- The level of difficulty of these problems should roughly correspond to the average level of difficulty of those which will appear on the exam. Of course, there may be problems on the final exam which are somewhat longer or more involved.
- Since electronic aids will not be allowed in the exam room, use these aids in your review only as a last resort.
- The answers to the problems below appear at the end.
- Problems 36-50 are not relevant for students in the two BioChem sections. The students in the BioChem sections should replace Problems 36-50 with the odd numbered problems from the chapters that those sections covered in the book by Rosner
- Students can obtain additional answered review problems by working the relevant odd numbered problems from Chapters 9.4-9.9, 10.1-10.5, 10.7, 11.1, 11.3, 12.1-12.10, 13.1-13.6, 14.1-14.8 and at the ends of Chapters 9-14 in Part II of the 9th edition of Thomas and Finney's book Calculus. The latter is on reserve in Cabot Library. Moreover, almost any book on multi-variable calculus will cover essentially the same subjects as we did here. Thus, even more problems for review can be had by working answered problems in other multivariable calculus books. (Please don't take such books out of Cabot; photocopy some problems instead so that other students can have access to the same resource.)

PROBLEMS:

1. Give an equation of the form f(x, y) = 0 for the following parametrized curves in $\mathbf{R}^2$:
   a) $x = (t^2 + 1)^{1/4}$, $y = 1 - t$.
   b) $x = 2\tan(t)$, $y = 1/\cos(t)$ for $-\pi/2 < t < \pi/2$.
   c) $x = 4\cos(t)$, $y = 3\sin(t)$.

2. In each case, give an equation for the line in R^2 which is tangent to the given curve at the indicated point:
   a) The curve is parametrized as $x = (t^2 + 1)^{1/4}$, $y = t$ and the point is where $t = 0$.
   b) The curve is where $x^3 + y^2 = -23$ and the point is (-3, 2).
   c) The curve is where $x + y^3 = 1$ and the point is (2, -1).

3. Find the length of the curve parametrized by $x = e^{2t} - 2t$, $y = 4e^t$ for $0 \le t \le 1$.

4. a) Find a parametrization of the form $t \to (x(t), y(t))$ for the curve in $\mathbf{R}^2$ which is parameterized in polar coordinates by $r = t$, $\theta = t^3$ with $t \ge 0$.
   b) Write this curve in the form f(x, y) = 0.

5. In each of the cases below, write the vector **B** as a sum of a vector which is parallel to the vector **A** and which is perpendicular to **A**.
   a) **A** = (1, 2, 2) and **B** = (1, 2, -1).
   b) **A** = (3, -4, 0) and **B** = (5, 1, 1).
   c) **A** = (2, -1, -2) and **B** = (3, 3, 3).

6. Suppose that **v** and **w** are vectors in R^3 with |**v**| = 2 and |**w**| = 3. In each case below |2**v** - **w**| is given. Decide whether **v** and **w** are perpendicular or not, or whether there is not enough information to decide.
   a) |2**v** - **w**| = 12.
   b) |2**v** - **w**| = 7.
   c) |2**v** - **w**| = 5.

7. Find the distance from the point (1, 2, 1) to the following:
   a) The plane where $2x + y - 2z = 0$.
   b) The line parameterized by $t \to (6t, 3t + 2, 2t + 1)$.

8. In each case, find an equation of the form f(x, y, z) = 0 for the indicated plane:
   a) The plane through the point (1, 0, 0) which is normal to **A** = (2, 1, -1).
   b) The plane containing the points (1, 1, -1), (2, 1, 0) and (3, 3, 3).
   c) The plane through the point (-1, 0, 0) for which **A** = (-1, -1, 1) and **B** = (1, 1, 3) are tangent.

9. Find the absolute value of the cosine of the angle between the planes $x = 5$ and $6x + 3y + 2z = 2$.

10. Write a parametric equation for the line through the origin which is normal to the plane through the three points (0, 1, 0), (1, -1, 1) and (1, 1, -1).

11. In each of the cases below, the given vector function of the parameter t is meant to be the velocity vector of a parametrized curve in $\mathbf{R}^3$. Decide whether the given curve lies entirely in a single plane.
   a) **v**(t) = (5 cos(t), 3 sin(t), cos(t)).
   b) **v**(t) = ($8t^2$, 3 t, cos(t)).
   c) **v**(t) = ($8t^2$, cos(t), -7 cos(t)).

12. In each case, find the linear approximations to the given function at the indicated points:
   a) $f(x, y, z) = 10x^2 + yz - z^2 + 1$; and the points are (1, 1, 1) and (1, 1, -1).
   b) $f(x, y, z) = \sin(xyz^2) + z$; and the points are (1, 2, 0) and (3, 0, 1).

13. In each case, find the equation for the tangent plane to the given surface at the indicated points:
   a) The surface is where $e^{xyz} - 2 + z = 0$ and the points are (0, 0, 1) and (0, 1, 1).
   b) Thus surface is where $x^2 + y^2 - xyz = 1$ and the points are (1, 0, 1) and (1, 1, 1).

14. Write down the linear approximation at (1, 1, 1) for any function f(x, y, z) on $\mathbf{R}^3$ with following properties:
   a) f(1, 1, 1) = -2
   b) Both **A** = (1, 1, 3) and **B** = (3, 1, -1) are tangent to the level set f = -2 at (1, 1, 1).
   c) The directional derivative of f in the direction (1, 0, 0) is 2.

15. Let $f(x, y, z) = x^2 - yz + 3$. In each case below, the given point lies on a parametrized curve and the given vector **v** is the tangent vector to that curve at the given point. Give the instantaneous rate of change of f along the curve at the given point.
   a) The point is (1, 1, 1) and **v** = (1, 0, 0).
   b) The point is (1, -1, 1) and **v** = (0, -1, 0).

c) The point is (0, 2, 2) and **v** = (1, 1, 0).

16. Suppose that f(x, y) is a function on $\mathbf{R}^2$ whose gradient at the origin is (1, -3). Also, suppose that g(u, v) = (x(u, v), y(u, v)) is a function from $\mathbf{R}^2$ to $\mathbf{R}^2$ which sends (0, 0) to (0, 0). Also, suppose the partial derivatives of x(u, v) and y(u, v) at (0, 0) are: $x_u = 1$, $y_u = 1$, $x_v = -1$, $y_v = 1$. Give the gradient vector at (0, 0) of the function G(u, v) = f(x(u, v), y(u, v)).

17. In each case, give the local maxima, then the local minima, and finally, the saddle points for the given function:
    a) $f(x, y) = y^2 + \cos(x)$.
    b) $f(x, y) = \cos(x) \sin(y)$.
    c) $f(x, y) = y^3 - 3y + x^2$.

18. Find the points where the function f(x, y) = 3xy + 1 takes on its maximum and minimum values on the region where $x^2/9 + y^2/4 \leq 1$.

19. Find the points of the ball $x^2 + y^2 + z^2 \leq 1$ where the function $f(x, y, z) = 2x^2 - y^2 - \cos(z^2)$ achieves its maximum and minimum values.

20. Find the point or points on the surface $x + y^2 + z^2 = 1$ where $f(x, y, z) = x/(1 + y^2 + z^2)$ has its maximum.

21. Version 1: Find the distance to the origin of the point on the surface where $x^2 + z^2y^2 = 1$ which is closest to the origin.

    Version 2: Find the distance to the origin of the point on the surface where $\frac{x^2}{4} + z^2 y^2 = 1$ which is closest to the origin.

22. Let $\mathbf{E}(x, y, z) = (x^2, zy + 1, 2y - z)$. Find the point or points in $\mathbf{R}^3$ where the length of **E** is least.

23. Find the area of the region where $x^2/4 + y^2/9 \leq 1$ and $x/2 + y/3 \geq 1$.

24. Find the average value of f(x, y) = x on the part of the first quadrant where $x^2 + y^2 \leq 1$.

25. Find the area of the region given in polar coordinates by $0 \leq r \leq \cos(\theta)$ and $0 \leq \theta \leq \pi/2$.

26. Integrate the function $f(x, y) = \cos(x^2)$ over the region in the first quadrant where $x - y \geq 0$ and $x \leq 1$.

27. Find the volume of the region in $\mathbf{R}^3$ where $y \geq 0$ and $y + x^2 + z^2 \leq 1$.

28. Find the volume of the region in $\mathbf{R}^3$ where $0 \leq z \leq \cos(x^2 + y^2)$ and $x^2 + y^2 \leq 1$.

29. Integrate the function $f(x, y, z) = \sin((x^2 + y^2 + z^2)^{3/2})$ over the interior of the ball of radius 1 in $\mathbf{R}^3$ whose center is at the origin.

30. Write in cylindrical coordinates the integral of the function $f(x, y, z) = y x^2$ over the region where $-1 \leq x \leq 1$, $0 \leq y \leq (1 - x^2)^{1/2}$ and $0 \leq z \leq (x^2 + y^2)^{1/2}$. Then, evaluate the resulting integral.

31. Write in spherical coordinates the integral of the function xyz^2 over the portion of the first quadrant where $(x^2 + y^2)^{1/2} \leq z \leq 1$. Then, evaluate the resulting integral.

32. Consider the inverted cone whose vertex lies at the origin and whose base is the disc in the plane $z = 3$ with center the origin and radius 4. Express the volume of the portion of this cone where $x^2 + y^2 + z^2 \leq 1$ as an iterated integral in rectangular, cylindrical and also spherical coordinates. Finally, find this volume by evaluating one of your integral expressions.

33. a) Write the integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dxdy$ in polar coordinates and evaluate.

b) Evaluate $\int_{-\infty}^{\infty} e^{-x^2}\,dx$. (Hint: Consider the square of this integral.)

34. Write down the function $u(t, x)$ which solves the equation $u_t = -4\, u_x + u$ and equals $(1 + x^2)^{-1}$ at $t = 0$.

35. A typical advection equation for a function $u(t, x)$ has the form $u_t = -c\, u_x + k$, where c is a constant and k is a function of t and x which might depend on u. On the other hand, a typical diffusion equation for a function $u(t, x)$ has the form $u_t = \mu\, u_{xx} + k$, where μ is a constant and where k is a function which might depend on u. In each of the following cases, indicate whether the function in question is more likely to obey an advection or a diffusion equation:

a) The function $u(t, x)$ gives the density of a heart stimulating drug in the blood as a function of time t and distance x along a vein leading to the heart. The drug is injected in the vein at $t = 0$.

b) The function $u(t, x)$ gives the density of cars on the west bound side of the Massachusetts Turnpike as a function of time t and distance x from the Boston end.

c) The function $u(t, x)$ gives the temperature of glacial ice in Greenland as a function of time t and distance x from the surface of the glacier.

- **The following review problems are not relevant for the BioChem sections.**

36. Compute the line integral of $\mathbf{F} = (y \sin(\pi z/4), x \cos(\pi z/4), -(y^2 + x^2) z^4)$ around the circle which is cut from the sphere $x^2 + y^2 + z^2 = 10$ by the plane $z = -3$. Traverse this circle counter-clockwise as viewed from the $z = 0$ plane.

37. Compute the line integral of the vector field $\mathbf{F} = (-\sin(\pi z), \cos(\pi z), -(y^2 + x^2) z^4)$ along the straight line segment starting at the origin and ending at (1, 1, 1).

38. Compute the line integral of the vector field $\mathbf{F} = (x y^4 \sin(\pi y^5/2), yx^2)$ along the path which traverses (counter-clockwise) the square in the x-y plane with vertices (0, 0), (1, 0), (0, 1) and (1, 1).

39. Parametrize the following surfaces:
    a) The portion of the surface where $x^2/4 + y^2/9 + z^2/25 = 1$ which lies where $x \geq 0$.
    b) The portion of the same surface which lies where $y \geq 0$.
    c) The portion of the surface where $x + y^2 + z^4 = 1$ which lies where $x \geq 0$.
    d) The portion of the plane where $x + y + z = 2$ which lies above the first quadrant in the x-y plane.

40. For Cases a-c in the previous problem, write down an iterated integral which computes the area of the indicated portion of the surface.

41. Find the average height over the x-y plane of the $z \geq 0$ part of the surface $x^2 + y^2 + z^2/25 = 1$.

42. Find the flux of $\mathbf{F} = (\sin(x^4 + z^3), \cos(x + z(y^2 - 1), z x^2)$ through the part of the $y = 1$ plane where both z and x are positive, but both are less than 1. Use the normal that has positive dot product with the position vector at the point (0, 1, 0).

43. Find the flux of $\mathbf{F} = (0, 0, z)$ through the surface $x^2 + y^2 + z^2 = 25$. Use the normal that points away from the origin.

44. Write down a vector field on $\mathbf{R}^3$ which is not constant, but has zero curl and zero divergence.

45. a) Write down a vector field on $\mathbf{R}^3$ which has divergence equal to xyz.
    b) Write down a vector field on $\mathbf{R}^3$ whose curl is equal to (1, 2, 3).

46. a) Write down a vector field on the plane whose counter-clockwise path integral around any circle equals the square of the radius of the circle.

b) Write down a vector field on $\mathbf{R}^3$ whose outward flux through the surface of any cube is equal to the third power of the length of an edge.

47. a) Either exhibit a vector field whose curl is (x, -2y, xy), or explain why no such vector field can exist.
    b) Either exhibit a vector field whose divergence is $y \cos(y z^2)$, or explain why no such vector field can exist.

48. Suppose **F** is a vector field whose curl is equal to (3, -5, 7). For each plane through the origin, one can consider the path integral of **F** in either direction around the circle where the plane intersects the sphere where $x^2 + y^2 + z^2 = 4$. Give the equation of a such a plane for which the path integral in question is zero.

49. Let f(x, y, z) be some (unspecified) function and let $g(x, y, z) = \int_0^x f(s, y, z)\, ds$. Give a formula which expresses the integral of the function f over the ball where $x^2 + y^2 + z^2 \leq 1$ as a surface integral over the boundary sphere which involves the function g. Use the divergence theorem to justify the validity of your formula.

50. In this problem, $\mathbf{F} = (-y, x)/(x^2 + y^2)$. This is a vector field which is defined everywhere on the plane except at the origin.
   a) Compute the counter-clockwise path integral of **F** around the circle $x^2 + y^2 = 1$.
   b) Suppose that γ_1 is a closed loop in the plane which encloses a region that contains the disk where $x^2 + y^2 \leq 1$. Compute the counter-clockwise path integral of **F** around γ_1.
   c) Suppose now that γ_2 is a closed loop which encloses a region which does not contain the origin. Compute the counter-clockwise path integral of **F** around γ_2.

   In all of the above, justify your answers.