Answers to the suggested problems for the Differential Equation lectures

Answers for the 12/11 suggested problems

1. a) $p(t) = p_0/(1 - c\, p_0\, t)$ where p_0 can be any constant.
   b) $p(t) = p_0/(1 - c\,(m - 1)\, p_0^{m-1}\, t)^{1/(m-1)}$ where p_0 can be any constant.

2. a) $p' = a\, e^t/(1 + a\, e^t) - a^2\, e^{2t}/(1 + a\, e^t)^2 = [a\, e^t/(1 + a\, e^t)]\,(1 - a\, e^t/(1 + a\, e^t))$.
   b) Find the constant a such that $p_0 = a/(1 + a)$. The answer is $a = p_0/(1 - p_0)$ and so the solution with $p(0) = p_0$ can be written as $p(t) = p_0\, e^t/(1 + p_0\,(e^t - 1))$.
   c) In this case, $f(x) = x\,(1 - x)$, so $f'(x_0) = 1$ and so the equation in question is $x' = x$, and the relevant solution is $x(t) = .001\, e^t$.
   d) In this case, $|p(t) - x(t)| = .001\, e^t\, |1/(1 + .001\,(e^t - 1)) - 1|$; and this last expression can be rewritten as $10^{-6}\, e^t\,(e^t - 1)/(1 + .001\,(e^t - 1)$, which is less than 10^{-5} when $t < 1$.

Answers for the 12/13 suggested problems

1. a) $u(t, x) = e^t \cos(x - 2\,t)$.
   b) $u(t, x) = e^t \sin(x - 2\,t)$.
   c) $u(t, x) = e^{t-1} \cos(x - 2\,t+2)$.
   d) $u(t, x) = e^{t-2} \cos(x - 2\,t+4)$.

2. If $u \equiv \frac{1}{t^{1/2}} e^{-(x-x_0)^2/4\mu t}$, then $u_t = -\frac{1}{2}\,\frac{1}{t^{3/2}} e^{-(x-x_0)^2/4\mu t} + \frac{(x-x_0)^2}{4\mu t^2}\,\frac{1}{t^{1/2}} e^{-(x-x_0)^2/4\mu t}$.
Meanwhile, $u_x = -\frac{(x-x_0)}{2\mu t}\,\frac{1}{t^{1/2}} e^{-(x-x_0)^2/4\mu t}$ and so $u_{xx} = -\frac{1}{2\mu}\,\frac{1}{t^{3/2}} e^{-(x-x_0)^2/4\mu t} + \frac{(x-x_0)^2}{4\mu^2 t^2}\,\frac{1}{t^{1/2}} e^{-(x-x_0)^2/4\mu t}$. Thus, $u_t = \mu\, u_{xx}$ as required.

3. If the entities are separated into two groups at time t = 0 and they don’t interact, then what happens to those in group 1 as time evolves does not depend on what happens to those in group 2. Thus, if u_1 describes the time evolution of the population in group 1 and u_2 that of the population in group 2, then the population for the two groups together should be the sum of those for the separate groups. Thus, $u = u_1 + u_2$ should solve the equation if u_1 and u_2 both do. Moreover, as the entities in group 1 don’t interact with each other, then changing the numbers in group 1 by a multiple should change the solution by the same multiple. Likewise for those in group 2. Thus, the solution whose time zero population is $a_1\, u_1(0) + a_2\, u_2(0)$ should be $a_1\, u_1(t) + a_2\, u_2(t)$.

Since $p' = c\,p$ is supposed to model the time evolution of a population of entities that don't interact, one should expect the superposition principle to hold. Since the general solution is $p(t) = p_0\, e^{ct}$ with p_0 a constant, the superposition principle does indeed hold here since u_1 must have the form $p_1\, e^{ct}$ and u_2 the form $p_2\, e^{ct}$ and so the combination $a_1 u_1 + a_2 u_2$ is equal to $(a_1 p_1 + a_2 p_2)\, e^{ct}$ which is, indeed, a solution to the exponential growth equation.

Answers for the 12/15 suggested problems

1. For $x < 0$, the integral in question can be written as

$$u(x) = -\,2^{-1} \int_0^{\infty} (s - x)\, e^{-s}\, ds = -\,2^{-1}\,(1 - x)\,.$$

Meanwhile, for $x \geq 0$, the integral is

$$u(x) = -\,2^{-1} \int_0^{x} (x - s)\, e^{-s}\, ds - 2^{-1} \int_x^{\infty} (s - x)\, e^{-s}\, ds = 2^{-1}\,(1 - x) - e^{-x}\,.$$

Notice that this function, even though defined by separate formulas for x positive and negative, is none-the-less continuous with a continuous derivative at $x = 0$.

2. $G_{xx} + G_{yy} + G_{zz} = \mathrm{div}(\nabla G)$. Now, $\nabla G = -\,\frac{1}{4\pi}\frac{\vec{r} - \vec{s}}{|\vec{r} - \vec{s}|^3}$. With this understood, write

$$\mathrm{div}(\nabla G) = \frac{1}{4\pi}\frac{1}{|\vec{r} - \vec{s}|^3}\,\mathrm{div}(\vec{r} - \vec{s}) - \frac{3}{4\pi}\frac{1}{|\vec{r} - \vec{s}|^5}(\vec{r} - \vec{s})\cdot(\vec{r} - \vec{s}) =$$

$$\frac{1}{4\pi}\frac{3}{|\vec{r} - \vec{s}|^3} - \frac{3}{4\pi}\frac{1}{|\vec{r} - \vec{s}|^3} = 0.$$

In this regard, remember that the product rule for derivatives implies that when f is a function and v is a vector field, then $\mathrm{div}(f\vec{v}) = f\,\mathrm{div}(\vec{v}) + \nabla f \cdot \vec{v}$. Also, remember that $\mathrm{div}(\vec{r}) = 3$.