Math 21a Supplement on Torque and Angular Momentum

According to Newton, the position vector, $\mathbf{r}(t)$, of a particle changes with time, t, under the influence of a force vector $\mathbf{F}$ according to the rule

$$m\ \mathbf{r}´´ = \mathbf{F}\ .$$
(1)

Here, m is the mass of the particle.

The <u>momentum</u> of the particle is, by definition, the vector $\mathbf{p} \equiv m\ \mathbf{r}´$. Thus, momentum is mass times velocity. The <u>angular momentum</u> is defined to be the vector

$$\mathbf{L} \equiv m\ \mathbf{r} \times \mathbf{r}´ = \mathbf{r} \times \mathbf{p}\ .$$
(2)

Note that $\mathbf{L}$ is perpendicular to both the position vector $\mathbf{r}$ and the velocity vector $\mathbf{r}´$.

In some sense, angular momentum measures the deviation from motion on a straight line. Indeed if $\mathbf{L} = 0$, then the velocity vector $\mathbf{r}´$ is proportional to the position vector $\mathbf{r}$, and this implies that the particle travels along a straight line (but maybe back and forth). To see that the condition $\mathbf{r}´$ being proportional to $\mathbf{r}$ means straight line motion, differentiate the unit vector $\mathbf{r}/|\mathbf{r}|$ under this assumption to see that the latter is constant.

In general, $\mathbf{L}$ will evolve in time if $\mathbf{r}$ does. Indeed, differentiate (2) to find that

$$\mathbf{L}´ = m\ \mathbf{r}´ \times \mathbf{r}´ + m\ \mathbf{r} \times \mathbf{r}´´ = m\ \mathbf{r} \times \mathbf{r}´´ = \mathbf{r} \times \mathbf{F}\ .$$
(3)

Here, the final equality comes by substituting (1) for $m\ \mathbf{r}´´$ while the second equality arises because $\mathbf{r}´ \times \mathbf{r}´ = 0$. The vector $\mathbf{r} \times \mathbf{F}$ is called the <u>torque</u>.

By the way, if $\mathbf{F}$ has the form $\mathbf{F} = f(r)\ \mathbf{r}/r$, then the angular momentum vector is constant since then $\mathbf{r} \times \mathbf{F} = 0$. An example is the case where the force is due to the gravitational pull of a mass, M, at the origin, for here, $\mathbf{F} = - G\ m\ M\ r^{-1}$, where G is the gravitational constant.

Finally, note that if $\mathbf{L}$ is constant, then the motion takes place solely in a plane. Indeed, this follows because

- $\mathbf{r}$ is always perpendicular to $\mathbf{L}$ as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.
- And, the vectors perpendicular to a given vector define a plane.

(4)

For example, the trajectory of a planet orbiting the sun lies in a plane. (This last statement is one of Kepler's laws. Kepler based this 'law' on empirical data from observations of the sky, and Newton got justly famous for using his laws of motion to explain why Kepler's laws hold.)