Math 21a Handout on Differential Equations

This handout introduces the subject of differential equations. But it barely scratches the surface of this vast, growing and extremely useful area of mathematics.

1. Differential equations in the sciences

The branch of mathematics called <u>differential</u> <u>equations</u> is a direct application of ideas from calculus, and as this is a mathematics course, I should begin by telling you a little bit about what is meant by the term ‘differential equation’. However, I’ll digress first to begin an argument for including mathematics in the tool kit of even the most experimentally minded scientist.

a) Modeling in the sciences

First, I freely admit to not being an experimentalist. In fact, until recently, I always found the theoretical side of science much more to my liking. Moreover, I suffered from a fairly common misconception:

If I only learn enough mathematics, I can uncover nature’s secrets by pure logical deduction.

I have lately come to the realization that advances in science are ultimately driven by knowledge dug from observations and experiments. Although logic and mathematics can say a great deal about the suite of possible realities, only observation and experimentation can uncover the detailed workings of our particular universe.

With the preceding understood, where is the place for mathematics in an experimentally driven science? The answer to this question necessarily requires an understanding of what modern mathematics is. In this regard, I should say that term ‘mathematics’ covers an extremely broad range of subjects. Even so, a unifying definition might be:

Mathematics consists of the study and development of methods for prediction.

Meanwhile, an experimentally driven science (such as physics or biology or chemistry) has, roughly, the following objective:

To find useful and verifiable descriptions and explanations of phenomena in the natural world.

To be useful, a description need be nothing more than a catalogue or index. But, an explanation is rarely useful without leading to verifiable *predictions*. It is here where mathematics can be a great help. In practice, experimental scientists use mathematics as a tool to facilitate the development of predictive explanations for observed phenomena. And, this is how you can profitably view the role of mathematics. (The use of mathematics as a tool to make predictions of natural phenomena is called modeling and the resulting predictive explanation is often called a mathematical model.)

At this point, it is important to realize that a vast range of mathematics has found applications in the sciences. One, in particular, is differential equations.

b) Equations

The preceding discussion about predictions is completely abstract, and so another digression may prove useful to bring the discussion a bit closer to the earth. In particular, consider what is meant by a prediction: You measure in your lab certain quantities---numbers really. Give these measured quantities letter names such as ‘a’, ‘b’, ‘c’, etc. A prediction can take the form of a formula which determines the value for the quantity c by measuring only the quantities a and b. Such a formula might involve simply an algebraic equation which relates a and b to c.

For example, if you lived in Greece some twenty five hundred years ago, you might discover that the length, c, of the hypotenuse of a right triangle can be predicted from the measured lengths, a and b, of the other two sides. Indeed, if you were Pythagoras, you would write:

$$c = \sqrt{a^2 + b^2} \tag{1.1}$$

Or, you might determine that the area, A, of a disk can be predicted with knowledge of its radius, r, using the equation

$$A = \pi r^2 . \tag{1.2}$$

These are examples of algebraic equations in that they involve simple expressions between what is known (a and b in (1.1)) and what is to be predicted (c in (1.1)). A famous and modern algebraic equation is Einstein’s formula

$$E = m c^2 \tag{1.3}$$

which describes how the total energy (E) of a body at rest can be computed if you know its mass (m) and the speed of light (c ≈ 3 million meters per second). A algebraic equation with applications to biology describes how the weight of a body (say w) would change if it had weight w_0 and you hypothetically scaled its length, width and height by the same factor, say s. This formula asserts that

$$w = s^3 w_0 .$$

(1.4)

c) Differential equations.

Differential equations can arise when studying quantities which depend on some auxiliary variable. For example, it is typical in the sciences to study time dependent phenomena. A doctor can be concerned with the amount of a certain medicinal drug in the body as a function of time. That is, there is a function which depends on the variable t = 'time' and its value at time t, say f(t), is the concentration of the medicine at time t in the blood.

Here is another example: An environmental scientist can be concerned with the concentration of mercury in clams along a certain stretch of river. Here, the concentration might depend on the distance downstream. Thus, the concern is with a function which depends on the variable x = 'distance downstream' and its value at distance x, say f(x), is the concentration of mercury in clams which are found at distance x. By the way, this concentration might depend on both position <u>and</u> time--a more complicated situation which shall also concern us.

Here is a third example: A developmental biologist studying fly embryos might be concerned with the level of a certain molecular growth factor as a function of distance from the embryo head. Here, the function in question is the level of the growth factor as a function of the variable which measures the distance from the head of the embryo. Of course, this function can also depend on time as well as position; and it most probably does since live embryos develop as time progresses.

For a fourth example, an epidemiologist might consider the number of deaths from a certain disease as a function of age at death. Here, the variable is the age, α, at death, and the number of deaths of people at age α from the disease gives the function. One could denote the latter by $N(\alpha)$. By the way, this example illustrates an important point: The variable in question need not be time nor a position, but some entirely different quantity. Indeed, the same epidemiologist might consider the average number of heart attack victims in a particular locale, as a function of the level of cholesterol in the victim. In this case, the variable, call it c, is the level of cholesterol, and the function in question assigns to each value of c the number n(c) which is the yearly average of heart attack sufferers with cholesterol level c.

With these examples understood, one can say that a <u>differential</u> <u>equation</u> for a function (or for some collection of functions) of a variable (or collection of variables) is simply an algebraic equation which involves both the function <u>and</u> its *derivatives*. For example, the equation

$$\frac{dp}{dt} = p \tag{1.5}$$

is a differential equation.

d) Continuity and differentiability

But for a few key exceptions, differential equations involve functions with derivatives, and thus functions which are apriori assumed to be continuous. However, it is important to realize that continuous functions may not be appropriate for describing certain natural processes. For example, the function which assigns to a time t the number N(t) of live bacteria in a petri dish is not a continuous function. Indeed, this function can only take values which are whole numbers, that is N(t) = 0, 1, 2, ... Thus, N(t) is either 3 or 4 but never the number π = 3.1416..., let alone 3.5.

None the less, it is sometimes a reasonable approximation to reality to pretend that N(t) is continuous. For example, if the number of bacteria is measured in units of 1 million, then N(t) can take on values which differ by .000001; in this case, the approximation that N(t) is continuous might not look so bad. In particular, if the experimental error in counting bacteria is greater than 1, then little is lost by modeling N(t) (here, measured in units of 1 bacteria) as a continuous function. Indeed, in this case, the discreteness of N is effectively invisible. In any event, be forewarned that the use of continuous functions in the sciences constitutes a modeling approximation which may or may not make sense in any given application.

In particular, here is a general rule of thumb which summarizes the preceding discussion:

- *If the true function under discussion jumps in value, then its replacement with a continuous function is reasonable when the experimental error is larger than any of the jumps.*

Of course, one can also consider whether the assumption of differentiability makes sense in any given situation. The rule of thumb here is usually the following:

- *Once the step to a continuous function is made, the step to differentiability rarely adds trauma.*

2. The simplest 1-variable differential equations

Suppose that you are interested in the values of some quantity, say p, as a function of time, t. A differential equation for p is an equation which equates the time derivative of p to some function of p.

a) The simplest equation

The simplest differential equation reads

$$\frac{d}{dt} p = 0 \tag{2.1}$$

which asserts that the quantity p is constant in time. A less trivial example would be

$$\frac{d}{dt} p = c ,$$

(2.2)

where c is some constant, say 2 or -3.4 or 10^7. (When this equation arises in the real world, the constant c is usually determined by some experimental measurement.) This last equation asserts that the rate of change of p is constant with time. Equation (2.2) can be solved fairly easily by integrating both sides:

$$p(t) - p(0) = \int_0^t \frac{dp}{dt} = ct .$$

(2.3)

The first equality here is just the Fundamental Theorem of Calculus. Thus, the solution to (2.2) has

$$p(t) = p(0) + ct .$$

(2.4)

b) Equations which involve p and its derivative

The simplest differential equation which involves both p and its derivative has the general form

$$\frac{d}{dt} p = f(p)$$

(2.5)

where f is some given function of p. This equation asserts that the rate of change of p at a given time is determined by the value of p at that time. A very important example is

$$\frac{d}{dt} p = p ,$$

(2.6)

and, more generally,

$$\frac{d}{dt} p = ap ,$$

(2.7)

where a is some given number, say, a = 3 or -5.56 etc.

These last two equations assert that the rate of change of p is proportional to p itself, the constant a being the proportionality factor. Here is a sample context in which (2.7) arises: Suppose that p(t) represents the number of bacteria in a petri dish at time t and that the dish is well sugared, so that the bacteria don't lack for food. One expects that each bacteria will fission into two bacteria at a regular rate, say once every 20 minutes. If no bacteria die, then the rate of change of p is equal to .05p in units of bacteria per minute. Put differently, if there are p(t) bacteria at time t, then one expects that roughly one out of every 20 bacteria are undergoing fission at any given minute. Thus, the population of bacteria in the petri dish at time t is governed by Equation (2.7) with a = .05 bacteria per minute.

c) The solution in the general case

There is always a solution to (2.5). Indeed, here is a general fact:

Choose any starting value for p(0) *and there exists a unique solution, p(t), to (2.5) which has the given starting value at t = 0.*

(2.8)

Unfortunately, for a complicated function f in (2.5), there will not be a closed form expression for the solution p(t). Even so, one can in principle find all solutions to (2.5) for a given choice of f. Here is how: First, introduce F(p) to denote an anti-derivative of the function 1/f(p). Then the solution p(t) to (2.5) obeys the following non-differential equation at all times t:

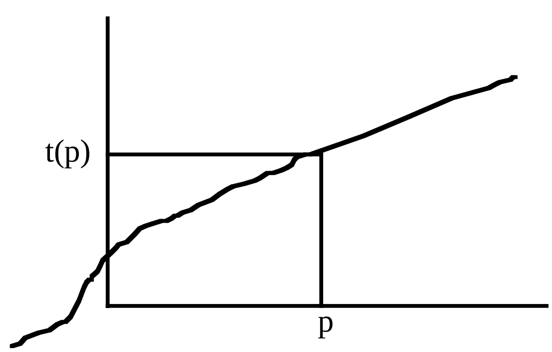
$$F(p(t)) = t + c \,. \tag{2.9}$$

Here, c is a constant which is determined by the chosen starting value for p(t). Indeed, if you substitute t = 0 in (2.9), you find that c = F(p(0)).

By the way, you can verify that (2.9) must hold by differentiating both sides with respect to t while using the Chain rule to write F(p(t))´ = p´(t)/f(p(t)).

Unfortunately, (2.9) says that t = F(p) which gives t as a function of p, rather than the desired p as a function of t. In principle, you can get from t as a function of p to p as a function of t, but if F is complicated, there won't be a closed form expression for the result.

As an aside, note that the derivation of p(t) from t(p) can be done graphically: First graph t as a function of p on paper with the vertical axis labeled t and the horizontal axis labeled p. Then, to find the value of p at a given time t, take the p value of the point where the horizontal line through (0, t) intersects the graph. The following figure gives an example of this technique:



(2.10)

Note that you need to be careful with your interpretation of the result when the horizontal line through (0, t) intersects the graph in more than one point!

d) The case where f(p) = ap

Probably the most important case of (2.5) is the case presented in (2.7) where f(p) = ap with a = constant. According to (2.9), the general solution to (2.7) is

$$p(t) = p(0)e^{at} . \tag{2.11}$$

Here, p(0) is the value of p at time 0. Alternately, you can write (2.11) as

$$p(t) = p(t_0)e^{a(t - t_0)} , \tag{2.12}$$

where t_0 is any convenient time and $p(t_0)$ is the value of p at that time. Do you believe that (2.11) and (2.12) are the same? Don't let me con you. Check that they are the same by using (2.11) to solve for $p(t_0)$ and then plugging the result into (2.12).

By the way, notice that when a > 0, the quantity p(t) is increasing with time, and when a < 0, then p(t) decreases with time. These are important features of (2.7) which play a key role in their applications.

The validity of (2.11) (or (2.12)) can be established by plugging the right hand side of (2.11) (or (2.12)) into (2.7) and differentiating. Here, you should remember that

$$\frac{d}{dt} e^{at} = a\, e^{at} . \tag{2.13}$$

Because (2.11) is an exponential of t, the equation in (2.7) is often called the <u>exponential growth equation</u>.

e) When does the exponential growth equation arise?

The exponential growth equation $\frac{d}{dt} p = ap$ occurs ubiquitously in the sciences. There are two reasons for this. Here is the first reason: Suppose that p(t) represents the <u>population</u> of identical particles or creatures at time t which <u>do not interact</u> with each other. Use a_b to denote the birth rate and a_d to denote the death rate. Then, the population p(t) will obey an exponential equation where the coefficient a is equal to the difference, $a_b - a_d$, (Note that if a_b is the birth rate (measured, say in births per second), then $1/a_b$ is the average time between births. Likewise, $1/a_d$ is the average time between deaths. For example, if the birth rate is 4 per/day, then, all things being equal, on average there will be a birth every quarter day.)

Notice that these quantities a_b and a_d, and hence the quantity a can, in principle, be determined by experiments on some small number of creatures (or particles) in isolation. This is what makes the exponential equation so useful. Experiments done with small numbers of creatures or particles can be used to predict the behavior of large numbers of particles. This is the great utility of (2.7) and (2.11). They allow you to predict the behavior of large numbers of creatures or particles in terms of quantities which have been measured from experiments with small numbers of particles. However, the large numbers of particles or creatures must not interact with each other in a substantive way. If they do, then all bets are off vis a` vis the applicability of (2.7) or (2.11).

The second reason for the ubiquitous appearance of (2.7) comes via a strategy which involves the replacement of a complicated function by its linear approximation near a point of interest.

To elaborate on this strategy, remark that functions can be arbitrarily complicated. For example, consider the function

$$f(x) = \cos(\sin(\frac{3}{\sin(x^3) + 4 + x\cos x})) \ . \tag{2.14}$$

What would you do with the differential equation

$$\frac{d}{dt}x = f(x) \tag{2.15}$$

for a function x of t, where f is given by (2.14)?

(Note that here the unknown function is x and not p nor even q. Remember that there is no universal ‘name’ for the unknown function and so I (and you) can name it what you like.)

The replacement of f by its linear approximation allows you to deal with a horrible function f in (2.15). However, be aware that there is a cost to making this replacement; I discuss the cost below.

The linear approximation replacement can be useful in the case that you are only interested in the solution x(t) to (2.15) only for x near some point x_0 of interest. If this happens to be the case (and often it is), then you only need to know about the function f near the point x_0. For example, x_0 might be zero, or it might be 12.33 or -21.677.

In any event, the strategy is to sacrifice some accuracy for solvability. You replace the horrible function f(x) with an approximate function, a function L(x) which is very close to f(x) as long as x is close to x_0. The function L(x) should be such that you can actually solve the equation

$$\frac{d}{dt}x = L(x) \ . \tag{2.16}$$

(Otherwise, why go through the trouble?)

Here is the key point:

The solution, x(t), *to (2.16) will behave much like the solution to (2.15) for those times t where x(t) from (2.16) is near to* x_0.

That is, you are interested in the solution to (2.15), but this equation is too hard to solve. So, instead, you solve an easier equation, (2.16), and observe that for times t where the solution to (2.16) is near to x_0, then the solution to (2.16) will provide a reasonable approximation to the desired solution to (2.15). Thus, you will gain some knowledge at the expense of some accuracy. The cost in making the replacement of (2.15) by the approximation, (2.16), is accuracy. The gain is (some) knowledge. Whether the gain is worth the cost depends on the circumstances.

With the preceding understood, consider using for L(x) in (2.16) the linear approximation to the given function f(x) in the vicinity of the point x_0 of interest. Thus,

$$L(x) = f(x_0) + f'(x_0)\,(x - x_0)\,. \tag{2.17}$$

In this case, (2.16) reads

$$\frac{d}{dt} x = f(x_0) + f'(x_0)\,(x - x_0)\,. \tag{2.18}$$

In the case where $f'(x_0) = 0$, this last equation is the same as equation (1) with $c = f(x_0)$, in which case the solution is

$$x(t) = x_0 + f(x_0)\, t\,. \tag{2.19}$$

In the case where $f'(x_0) \neq 0$, the equation in (2.18) is, after a change of variables, just the exponential equation from (2.7) with $a = f'(x_0)$. Indeed, introduce

$$u(t) = x(t) - x_0 + f(x_0)/f'(x_0). \tag{2.20}$$

Then, the right hand side of (2.18) is just $f'(x_0)\, u$. Meanwhile, as x_0 and $f(x_0)/f'(x_0)$ are constants, the time derivative of u(t) is the same as that of x(t). And, with this understood, one sees (2.18) as being equivalent to

$$\frac{d}{dt} u = f'(x_0)\, u\,. \tag{2.21}$$

This is just the exponential equation, with solution $u = u(0)\, e^{f'(x_0)\, t}$. Now, use (2.20) to write u in terms of x and so discover that

$$x(t) = x_0 + (e^{f'(x_0)\, t} - 1)\, f(x_0)/f'(x_0)\,. \tag{2.22}$$

I want to stress that this is not a solution to the original equation (2.16). However, as long as t is small (so that x(t) is nearly x_0), the function x(t) in (2.22) behaves very much like the solution to (2.16) which starts at x_0 when t = 0.

3. Introduction to advection

This section considers equations which can be used to predict the behavior of quantities which depend on both time and space. Here is an example for the sort of analysis that lies ahead: Suppose that the wind is blowing from west to east at a constant velocity of 3 meters per second. Meanwhile, an explosion in a chemical warehouse has pumped particulate pollution into the air. These particles then fall out of the air at a constant percentage rate r. What do we need to know to predict the particle concentration east and west of the explosion as a function of time and distance from the explosion?

a) What comes in must go out

Let u(t, x) be the function which gives the density (number of particles per meter) of particulate matter in the air at time t and point x along the west/east line through the warehouse. (Make the origin at the warehouse.) The following considerations lead to an equation for u: First, fix a point x and a small distance Δx. The amount of particulate matter in the region between x and x +Δx at time t is given (approximately) by

$$u(t, x)\, \Delta x \quad . \tag{3.1}$$

(This approximation becomes more and more accurate as Δx shrinks towards zero.)

The rate of change of (3.1) with respect to time is

$$\frac{d}{dt}(u(t, x)\, \Delta x) = q(t, x) - q(t, x + \Delta x) - k(t, x)\, \Delta x \quad , \tag{3.2}$$

where

- q(t, x) *is the number of particles per second which pass* x *from left to right minus the number or particles per second which pass* x *from right to left at time* t.
- q(t, x +Δx) *is the number of particles per second which pass* x + Δx *from left to right minus the number of particles per second which pass* x + Δx *from right to left at time* t.
- k(t, x) Δx *is the approximate number of particles which are created in the region between* x *and* x + Δx *at time t minus the number which are destroyed in this same region at time* t. *(In the example at hand,* k(t, x) = -r u(t, x), *but one can imagine more complicated terms here.)*

(3.3)

It is important to realize that (3.2) expresses nothing more than the tautology that the rate of change of the number of particles in the region between x and x + Δx is given by:

- *Adding the number of particles which enter across* x *and subtracting the number which leave across* x *(the term* q(t, x)*).*
- *Subtracting the number which leave across* x + Δx *and adding the number which enter across*
  x + Δx *(the term* q(t, x + Δx)*).*
- *Finally, adding the number which are created and subtracting the number which are destroyed in the region between* x *and* x + Δx *(this is given by the term* k(t, x) Δx*).*

If we divide both sides of (3.2) by Δx and take the limit as Δx tends to zero, we obtain

$$\frac{\partial}{\partial t}u(t, x) = -\frac{\partial}{\partial x}q(t, x) + k(t, x)\ . \tag{3.4}$$

In this regard, remember that

$$\frac{\partial}{\partial x}q(t, x) = \lim_{\Delta x \to 0}\ \frac{q(t, x + \Delta x) - q(t, x)}{\Delta x}\ ; \tag{3.5}$$

this explains the first term on the right hand side of (3.4).

b) The form of q

Equation (3.4) is not very useful unless we can find a reasonable form for both q(t, x) and k(t, x). Now in the example of the explosion, we have already decided to take k(t, x) = -r u(t, x) to account for the fact that the particles are lost at a constant rate r.

In our explosion example, there is also a relatively simple form for q(t, x) if we assume that the motion of the particles is entirely due to their being pushed along by the wind. In this case, the number of particles which pass the point x from left to right at time t is given by 3u(t, x) particles per second; and the number of particles which pass from right to left is equal to zero. That is, if the wind is blowing from left to right at a speed of 3 meters per second, then the number of particles per second which pass the point x at time t is given by the density (in units of particles per meter) times the wind speed (in units of meters per second). To summarize, under the preceding assumptions, one should take

$$q(t, x) = 3u(t, x)\ . \tag{3.6}$$

Please take note of the assumption that we used to derive (3.6): The particles are moving only because of the motion of the wind. This is a reasonably valid assumption if the particles are heavy, but if the particles are very light, then one expects some random dissipative motion even without the wind blowing. (We will see subsequently how to model the dissipative case too.)

c) The advection equation

If we plug k = -r u and q = 3u into Equation (3.4), we find that the function u(t, x) is predicted to be a solution to the advection equation

$$\frac{\partial}{\partial t}u(t, x) = -3\frac{\partial}{\partial x}u(t, x) - r\,u(t, x)\ . \tag{3.7}$$

So, with the preceding assumption about the form for q(t, x), the density function u(t, x) (which is what we are interested in) is constrained in the sense that its partial derivatives in the t and the x directions are related according to (3.7). Thus, if we know the solutions to (3.7), then we know something about the form of our mysterious function u. Equation (3.7) is our first example of what is often called a partial differential equation. The use of the word ‘partial’ is in reference to the partial derivatives which appear in the equation.

FACT: *Every solution to this equation can be written as*

$$u(t, x) = e^{-rt} f(x - 3t) \tag{3.8}$$

where f(x - 3t) means the following: Take any function f of one variable. Then, create the function f(x - 3t) of the variables t and x which is obtained by evaluating your original function f at the point x - 3t. Note that there is a huge set of solutions to (3.7), for I can choose any 1-variable function f to use in (3.8).

Here is how to prove that (3.8) solves (3.7): Simply compute the indicated partial derivatives using the Chain rule.

d) Initial conditions

There are infinitely many solutions of (3.7), one for each choice of 1-variable function f in (3.8). How does one determine the precise function f to use in (3.8)? Here is where the initial conditions enter. The term ‘initial condition’ signifies the value of u(t, x) as x varies but t is fixed at some predetermined time (say t = 0).

For example, suppose that the value of u at t = 0 has been determined apriori to be given as a function of x. Call this new function g(x). That is, suppose that knowledge of g(x), which is the time zero density, has been given. Then, there is one and only one solution to (3.7) with u(0, x) = g(x). This is the solution to (3.7) where the function f is precisely the function g. That is,

$$u(t, x) = e^{-rt} g(x - 3t) \tag{3.9}$$

is the <u>only</u> solution to (3.7) which <u>obeys</u> the initial condition

$$u(0, x) = g(x) . \tag{3.10}$$

The following summarizes this business with initial conditions:

The values of any solution u *to (3.7) at all times* t *and at all points* x *are predetermined by the time* 0 *values of* u *at all points* x.

(3.11)

In the explosion scenario above, one can imagine that a satellite photo has been taken at time 0 (a few minutes after the explosion), and that the particle density g(x) has been determined as a function of x at time 0 from the satellite photograph. Then, the values of u(t, x) at all subsequent times and all points can be predicted.

The point here is that the values $u(t_0, x)$ as x varies at fixed time $t = t_0$ completely determine the solution u(t, x) to (3.7).

e) Traveling waves

To get a feeling for what these solutions look like, consider (3.7) in the case where r = 0. As asserted above, the general solution has the form u(t, x) = f(x - 3t), where f(·) can be any function of 1-variable and u(t, x) is obtained from f by evaluating the latter at the point x - 3t. Now, what does this say about u? Among other things, it says that the value of u at a point (t, x) is the same as that of u at time 0, but not at x, rather at 3t units to the left of x. Said differently, if you were to walk to the right (increasing x) at speed 3 (so your x coordinate increases by the amount 3t after time t), then you would not see any change in the value of u. In this sense, the solution u(t, x) to the r = 0 version of (3.7) describes a concentration of particles which moves at speed 3 to the right, but otherwise maintains its shape. Likewise, the versions of (3.7) with $r \neq 0$ describe concentrations of particles which move at speed 3 to the right and either decrease ($r < 0$) or increase ($r > 0$) in time as they move.

f) Lessons

Here are some of the important lessons from this section of the handout:

- There is a tautological differential equation, (3.4), which describe the time and space dependence of the density of particles moving in a fluid. Here, q(t, x) takes into account the net flow of particles past x at time t, while k(t, x) takes into account the net number of particles created and destroyed at x at time t.
- In the case where particle motion is due to the constant velocity flow of the ambient fluid, then q = -c u, where $c > 0$ if the flow is from left to right on the x-axis, and otherwise $c < 0$. With this choice of q, (3.4) is called an advection equation.

- The advection equation in (3.7) is predictive in the sense that you can specify what you want for the function u at time t = 0 as a function of x and then there is a unique solution to the equation which is your specified function of x at t = 0.
- A solution to (3.7) resembles a traveling wave.

4. Diffusion equations

As in the previous section, the central issue here is that of predicting the behavior of a function which measures the density of some type of particle as a function of time and position. At the beginning of the previous section, I derived a completely tautological equation for this function. Let me remind you: I use t to denote the time coordinate and x to denote the space coordinate; and then I use u to denote the function. Thus, u(t, x) measures the density of particles at time t and position x along some line. By keeping track of all possible fates of particles which start near x at time t, I argued that u should obey the equation

$$\frac{\partial}{\partial t}u(t, x) = -\frac{\partial}{\partial x}q(t, x) + k(t, x)\ . \tag{4.1}$$

Here,

- q(t, x) = (the number of particles per second crossing x from left to right at time t) - (the number of particles per second crossing x from right to left at time t) ,
- k(t, x) = (the number of particles per second per unit length which appear at x at time t) - (the number of particles per second per unit length which disappear at x at time t) .

(4.2)

With regard to (4.1), don’t be misled by its stylish form, for (4.1) is only a fancy way of saying: “What comes in either stays in or goes out again”. However, this equation can become useful when the functions q and k are specified.

For example, in the last section, I considered the case where the particle motion is due entirely to the movement of the ambient fluid. Under this last assumption, I argued that the appropriate choice for q(t, x) = c(t, x) u(t, x), where c(t, x) is the velocity of the fluid at time t and position x. (In the last section, I took c to be a constant, but that extra simplification is often a poor approximation to reality.) In the previous section, I also took k(t, x) to equal -r u(t, x), where r is a constant. The result of these choices yields the advection equation version of (4.1),

$$\frac{\partial}{\partial t}u(t, x) = -\,c\frac{\partial}{\partial x}u(t, x) - r\,u(t, x)\ , \tag{4.3}$$

whose general solution has the form

$$u(t, x) = e^{-rt} f(x - ct) \tag{4.4}$$

with f(·) any function you like of a single variable.

I can't stress enough that (4.3) is appropriate only when the particle motion is, to a good approximation, due only to the motion of the ambient fluid. This may not be the case. For example, the fluid may be at rest, and the particles might, individually have random motions. In the latter case, a different choice for the function q(t, x) is correct and the resulting version of (4.1) is called a <u>diffusion</u> <u>equation</u>.

a) The diffusion equation

The archetypal diffusion equation has the form

$$\frac{\partial}{\partial t}u = \mu \frac{\partial^2}{\partial x^2}u + k(t, x) \tag{4.5}$$

where k(t, x) is a function which can depend on u(t, x); and where μ is a constant which is determined by the average speed of the particles. Here, $\frac{\partial^2}{\partial x^2}u$ signifies the second derivative of the

function u along the x direction. (So, keep t fixed as if it were a constant and pretend that you are taking the derivatives of a function of x only.)

More precisely, the constant μ is determined by the square root of the average of the square of the velocity of the particles. The derivation of the number μ from first principles requires some fairly sophisticated probability theory. In practice, the constant μ can be measured in the laboratory using some fairly simple experiments. The constant μ is called the <u>diffusion coefficient</u>.

b) The derivation of the diffusion equation

Equation (4.5) is of the form of (4.1) with $q(t, x) = -\mu \frac{\partial}{\partial x}u(t, x)$. This general form for q can be justified in a heuristic sense as follows: When the motion of the particles is completely random, one expects that the number of particles which cross x per second at time t from left to right is proportional to the number of particles which are just to the left of x at time t. If the velocities of the particles are truly random, then roughly half of these particles will be moving to the right (and so not cross x), while half will be moving to the left, and so will cross x. Meanwhile, the number of particles per second which cross x from right to left at time t should be proportional to the number of particles which are just to the right of x at time t. Thus, we expect q(t, x) to be positive if there are more particles just to the left of x at time t then just to the right, and we expect q(t, x) to be negative if the converse is true. And, we expect q(t, x) to vanish if the number of particles just to the right is the same as the number which is just to the left. But, this is essentially saying that q(t, x) should be proportional to $-\frac{\partial}{\partial x}u(t, x)$, since $-\frac{\partial}{\partial x}u(t, x)$ is positive if there are more particles just to the left of x, and negative if there are more particles just to the right, and zero otherwise.

c) A fundamental solution

As with (4.3), the diffusion equation in (4.5) for any given form for k(t, x) has a whole raft of solutions. Of particular importance is the function

$$u(t, x) = \frac{a}{t^{1/2}} e^{-x^2/4\mu t} \tag{4.6}$$

which solves the k(t, x) = 0 version of (4.5):

$$\frac{\partial}{\partial t}u = \mu \frac{\partial^2}{\partial x^2} u \, . \tag{4.7}$$

In (4.6), a is any constant.

For any fixed t, the graph of u(t, x) in (4.6) is the standard "bell shaped curve", but as t increases, the curve gets flatter and flatter. (Its maximum height is at the origin, where u(0, t) $=a/t^{1/2}$. Thus, as t increases, the particle distribution does indeed diffuse out from a concentrated peak at small t to an almost uniformly small density at large t. Try sketching the graph of u(1, x), u(2, x) and u(3, x) where u(t, x) is given by (4.6) with a = 1 and μ = 1/4. Here, use the same set of axis for each case, but use different colors so that you can distinguish the three different graphs.

By the way, at any fixed time t, the integral of u(t, x) over the whole of the x axis is constant. This is predicted by (4.7) since the term k(t, x) which models the appearance and disappearance of particles is missing.

d) The diffusion equation is predictive.

Somewhat more general then the k = 0 version of (4.5) takes k(t, x) = -r u(t, x). In this case, (4.5) reads

$$\frac{\partial}{\partial t}u = \mu \frac{\partial^2}{\partial x^2} u - r\, u \tag{4.8}$$

It is important to realize that the diffusion equation in (4.8), like the advection equation in (4.3), is <u>predictive</u> in the following sense:

Specify desired t = 0 *values for* u *as a function of* x*, and there is a unique solution* u(t, x) *to (4.8) which has these given values at time* t = 0. *In particular, every solution to (4.8) is determined for all points* x *at all times* t > 0 *by its* t = 0 *values.*

This property of being predictive is what makes the diffusion equation and the advection equation so useful.

(The predictive nature of these equations can be proved by making rigorous the following argument: If you know u(0, x), then the equation tells you what $\frac{\partial}{\partial t}u$ is at t = 0 and at any x. This tells you (approximately) u(Δt, x) for small Δt and all x. (But, accuracy increases as Δt goes to zero.) Plug your new found knowledge of u(Δt, x) into the left side of (1) or (2) to learn what $\frac{\partial}{\partial t}u$ is at t = Δt and any x. This tells you (approximately) u(2Δt, x) for small Δt and all x. Plug your new found knowledge of u(2Δt, x) into the left side of (1) or (2) to learn what $\frac{\partial}{\partial t}u$ is at the next time step, t = 2Δt and any x, and so on.)

For the advection equation, we saw already that the solution u(t, x) is determined by its t = 0 values via the formula

$$u(t, x) = e^{-rt}\, u(0, x - ct) \quad . \tag{4.9}$$

Here, knowing u(0, x) for all x explicitly allows you to compute u(t, x) for any t and x.

For the diffusion equation in (4.8), the dependence of u(t, x) for t ≥ 0 on the values of u at t = 0 has a more complicated form. In any event, here it is:

$$u(t, x) = \frac{1}{(4\pi\mu t)^{1/2}}\, e^{-rt} \int_{-\infty}^{\infty} u(0, s)\, e^{-(x-s)^2/4\mu t}\, ds. \tag{4.10}$$

In this last equation, you should think of the function $e^{-(x-s)^2/4\mu t}$ as a function of three variables, x, s and t. Then, (4.10) integrates over the s variable while keeping t and x fixed to obtain a function of x and t only.

e) Superposition

Both (4.3) and (4.8) obey the superposition principle. This principle goes as follows: Suppose that $u_1(t, x)$ and $u_2(t, x)$ are both solutions to either (4.3) or (4.8) and that a_1 and a_2 are numbers. Then

$$u(t, x) = a_1\, u_1(t, x) + a_2\, u_2(t, x)$$

(4.11)

is a solution to (4.3) or (4.8) as the case may be. This superposition principle is used over and over again in applications of these equations. (Equations for which the superposition principle holds are called linear equations. Generally, the superposition principle will fail if the equation involves powers of u greater than 1, or a function of u multiplying u or its derivatives or functions of derivatives of u multiplying derivatives of u. For example, the equation $\frac{\partial}{\partial t}u =$

$\mu \frac{\partial^2}{\partial x^2}u + r\, u^2$

does not obey the superposition principle.)

Argue as follows to check the superposition principle for (4.3):

Step 1: Since a_1 and a_2 are constants, one computes

$$\frac{\partial}{\partial t}u = \frac{\partial}{\partial t}(a_1u_1 + a_2u_2) = a_1(\frac{\partial}{\partial t}u_1) + a_2(\frac{\partial}{\partial t}u_2) \quad . \tag{4.12}$$

This is because the derivative of a sum of function is the sum of the derivatives, and the derivative of a real number times a function is the real number times the derivative of the function.

Step 2: Likewise, compute

$$\frac{\partial}{\partial x}u = \frac{\partial}{\partial x}(a_1u_1 + a_2u_2) = a_1(\frac{\partial}{\partial x}u_1) + a_2(\frac{\partial}{\partial x}u_2) \quad . \tag{4.13}$$

and multiply this last expression by -c to get

$$-c\frac{\partial}{\partial x}u = -c\frac{\partial}{\partial x}(a_1u_1 + a_2u_2) = a_1(-c\frac{\partial}{\partial x}u_1) + a_2(-c\frac{\partial}{\partial x}u_2). \tag{4.14}$$

Step 3: Compute

$$r\,u = r\,(a_1u_1 + a_2u_2) = a_1(r\,u_1) + a_2\,(r\,u_2) \quad . \tag{4.15}$$

Step 4: Add the right sides of (4.14) and (4.15), and likewise add the left sides of these equations. Since the right side of each is equal to the left side of each, you will find that

$$-c\,\frac{\partial}{\partial x}u + r\,u = a_1(-c\frac{\partial}{\partial x}u_1 + r\,u_1) + -a_2(-c\frac{\partial}{\partial x}u_2 + r\,u_2). \tag{4.16}$$

Step 5: Finally, because u_1 and u_2 satisfy (4.3), the right hand side of (4.16) is equal to

$$a_1\left(\frac{\partial}{\partial t}u_1\right) + a_2\left(\frac{\partial}{\partial t}u_2\right) \tag{4.17}$$

which is the right side of (4.12). Thus, the right side of (4.17) is equal to $\frac{\partial}{\partial t}u$ as it has to be if u is to satisfy (4.13) as claimed.

Please convince yourself of the validity of the superposition principle for (4.8).

This superposition principle is extremely important. For example, the basic solution to (4.7) is

$$u_0(t, x) = \frac{1}{t^{1/2}} e^{-x^2/4\mu t} \tag{4.18}$$

or, more generally, if x_0 is any given point, then

$$u_{x_0}(t, x) = \frac{1}{t^{1/2}} e^{-(x-x_0)^2/4\mu t} \tag{4.19}$$

is a solution to (4.7). Using the superposition principle, I can immediately conclude that I can select any number of points $\{x_0, x_1, \ldots\}$ and a like number $\{a_0, a_1, \ldots\}$ of numbers (i.e. $x_0 = 12$, $x_1 = -2.4, \ldots$, and $a_0 = 5$, $a_1 = -.3, \ldots$) and then

$$a_0 \frac{1}{t^{1/2}} e^{-(x-x_0)^2/4\mu t} + a_1 \frac{1}{t^{1/2}} e^{-(x-x_1)^2/4\mu t} \ \ldots \tag{4.20}$$

is also a solution to (4.7).

d) Lessons

There are three important lessons to be had from this lecture:

- Under the assumption that the particles are moving randomly, (4.1) becomes (4.5), the diffusion equation
- Both (4.3) and (4.7)) are predictive in the following sense: Pick any function of the variable x, call it $u_0(x)$, and then both equations have a unique solution u(t, x) which equals your chosen function at t = 0. That is, $u(0, x) = u_0(x)$.
- Both equations obey the superposition principle. This is to say that if you have two solutions and you multiply each by a chosen number and add the resulting functions together, then the function of t and x so obtained also solves the equation.

5. Laplace's equation

The previous section saw the derivation of the diffusion equation for a function of time and one space coordinate, x. There is an analogous diffusion equation for functions of time and two or three space coordinates which is derived by the same sort of bookkeeping considerations. Indeed, the diffusion equation for a function, u(t, x, y, z) of time and the usual three space coordinates reads:

$$\frac{\partial}{\partial t}u = \mu\,\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u + k(t, x, y, z)\ . \tag{5.1}$$

The case where there are only two space coordinates, x and y, is obtained from (5.1) by deleting all references to the coordinate z.

An important solution to the k = 0 version of the $\mathbb{R}^3$ version is

$$u(t, \vec{r}) = \frac{a}{t^{3/2}}\, e^{-|\vec{r}|^2/4\mu t}\ , \tag{5.2}$$

where a can be any constant. There is an analogous solution to the $\mathbb{R}^2$ version; it is given by (5.2) with t replacing $t^{3/2}$ in the fraction out front. In general, there are many solutions to (5.1) no matter the form of the function k.

By the way, the combination

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \tag{5.3}$$

of second derivatives appears so often in scientific applications that it is given a special symbol, usually Δ, although sometimes ∇^2. Note that Δu is the same as $\mathrm{div}(\vec{\nabla} u)$. The combination Δ of derivatives in (5.3) is called the Laplacian.

a) The time independent case

An important equation in its own right is the diffusion equation in (1.1) for a function on $\mathbb{R}^3$ or $\mathbb{R}^2$ which is independent of time. In this case, the function k is also assumed to be independent of time. Thus, one is looking for u(x, y, z) (or u(x, y)) which solves the equation

$$-\Delta u = k\ . \tag{5.4}$$

This last equation is called Laplace's equation; it arises in many scientific applications, even those with no 'diffusion' involved. For example, if k is interpreted as a density of electric charge in space, then the gradient, ∇u, gives the resulting electric field in space.

b) The Green's function

In the case where u and k are functions of just one coordinate, x, then (5.4) reads

$$- u_{xx} = k \,. \tag{5.5}$$

This last equation can be solved for u by integrating twice. For example:

$$u(x) = \int_0^x \left(\int_0^s k(\tau)\, d\tau \right) ds \tag{5.6}$$

However, you might be surprised to learn that there is a way of writing a solution to (5.5) which involves only a single integral (even though (5.5) involves two derivatives). In particular,

$$u(x) = - 2^{-1} \int_{-\infty}^{\infty} |x - s|\, k(s)\, ds \tag{5.7}$$

is a solution to (5.5) when the function k is zero outside of some bounded region on $\mathbb{R}$ (or else $|k(s)|$ tends to zero sufficiently rapidly as $|s| \to \infty$). It is a real test of your understanding of differentiation to take two derivatives of (5.7) and so verify that (5.5) is obeyed. By the way, note that the expression $2^{-1} |x - s|$ which appears in (5.7) is, for each fixed s, a bonafide function, G(x), on $\mathbb{R}$. Moreover, as $|x - s| = s - x$ where $x < s$ and $|x - s| = x - s$ where $x > s$, this function obeys the equation $G_{xx} = 0$ except at the point $x = s$ where its second derivative does not exist.

In the $\mathbb{R}^3$ and $\mathbb{R}^2$ cases of (5.4), there are analogous integration formulas which provide solutions to (5.4). In particular, if $k(\vec{r})$ is a function on $\mathbb{R}^3$ which is zero outside of some bounded region (or which gets small fast enough as $|\vec{r}|$ tends to infinity) then there is a unique solution, $u(\vec{r})$, to (5.4) which has the property that $|u(\vec{r})|$ also tends to zero as $|\vec{r}|$ tends to infinity. This solution is determined by the function k via a multiple integral:

$$u(\vec{r}) = \frac{1}{4\pi} \iiint \frac{k(\vec{s})}{|\vec{r} - \vec{s}|}\, ds_1\, ds_2\, ds_3 \,. \tag{5.8}$$

The expression $\frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{s}|}$ which appears above is called the <u>Green's</u> <u>function</u> for the Laplacian. Note that with $\vec{s}$ fixed, this Green's function is a bonafide function of points $\vec{r}$ in $\mathbb{R}^3$ except at the point $\vec{r} = \vec{s}$ where it blows up. It is left to you as an exercise to verify that this function, $G(\vec{r})$, satisfies $\Delta G = 0$ where $\vec{r} \neq \vec{s}$.

There is also a version of (5.8) for the case where u and k are functions on $\mathbb{R}^2$ instead of on $\mathbb{R}^3$. In this case, there is a solution, $u(\vec{r})$, to (5.4) which is given by the double integral

$$u(\vec{s}) = \frac{1}{2\pi} \iint k(\vec{s}) \ln(|\vec{r} - \vec{s}|)\, ds_1\, ds_2 \,.$$

(5.9)

In this two dimensional case, $\frac{1}{2\pi}\ln(|\vec{r} - \vec{s}|)$ is called the Green's function. Again, for fixed $\vec{s}$, this is a bonafide function, $G(\vec{r})$, of $\vec{r}$ except at $\vec{r} = \vec{s}$. And, as in the 3-dimensional case, $\Delta G = 0$ where $\vec{r} \neq \vec{s}$.

PROBLEMS:

1. Find, by integration, all solutions to the following differential equatons:
   a) $p' = e^p$.
   b) $p' = e^p + 1$.
   c) $p' = p\,(1 - p)$.

2. Find 3 real systems (in biology, economics, physics or what ever) where the exponential rowth equation $p' = a\,p$ is a reasonable model. Give a sentence of justification using either the birth/death model in the reading, or via the linear approximation, or through some other mechanism.

3. Find the solution, $u(t, x)$, to

$$\frac{\partial}{\partial t}u(t, x) = -3\frac{\partial}{\partial x}u(t, x) - 2\,u(t, x)$$

   which at $t = 0$ is
   a) e^{-4x}
   b) e^{-x^2}.

4. Consider the solution $u(t, x) = f(x - 3t)$ to the equation $\frac{\partial}{\partial t}u(t, x) = -3\frac{\partial}{\partial x}u(t, x)$ with the initial condition $u(0, x) = (1 + x^2)^{-1}$. On graph paper, sketch in different colors the graph of the function $y(x)$ in the cases where
   a) $y(x) = u(0, x)$.
   b) $y(x) = u(1, x)$.
   c) $y(x) = u(2, x)$.

5. Let $N(t, x)$ denote the number of people in Boston at time t and age x, both measured in the same units. Explain why this function might be expected to obey an advection equation of the form

$$\frac{\partial}{\partial t}N(t, x) = -\frac{\partial}{\partial x}N(t, x) - r(x)\,N(t, x).$$

for some suitable function r(x) of age. Indicate in your answer how one should interpret this function r(x) and sketch the r versus x graph of a reasonable possibility. Also, make sure to explain why the speed c in this equation has value +1.

6. Verify by taking the appropriate partial derivatives that $u(t, x) = \frac{a}{t^{1/2}} e^{-rt} e^{-x^2/4\mu t}$ is a solution to

$$\frac{\partial}{\partial t}u = \mu \frac{\partial^2}{\partial x^2}u - r\,u \qquad (*)$$

when μ and r are constants. Here, a is also a constant. (Note that r comes here with a + sign.)

7. Verify that the following functions are also solutions to Equation (*) in the previous problem:
   a) $e^{\lambda t}\; e^{((\lambda-r)/\mu)^{1/2} x}$ where λ is any constant with $\lambda > r$.
   b) $e^{\lambda t}\; e^{-((\lambda-r)/\mu)^{1/2} x}$ where λ is any constant with $\lambda > r$.
   c) $e^{rt}(a + bx)$ where a and b are any constants.
   d) $e^{\lambda t}\cos[((r - \lambda)/\mu)^{1/2}x]$ where λ is any constant with $\lambda < r$.
   e) $e^{\lambda t}\sin[((r - \lambda)/\mu)^{1/2}x]$ where λ is any constant with $\lambda < r$.

8. If a diffusion equation such as that in Equation (*) of Problem 6 is to lead to reasonable predictions of a given phenomena, it is important to discern the appropriate choices for the constants r and μ which appear in the equation. In this regard, it turn out that the values of almost any particular solution u(t, x) to Equation (*) at two fixed times, say t = 1 and t = 2, and two thoughtful choices for x can be used to determine a, r and μ. For example, as you verified
   in Problem 4, above, the function $u(t, x) = \frac{a}{t^{1/2}} e^{-rt} e^{-x^2/4\mu t}$ solves Equation (*); and the values of u(1, x) and u(2, x) for two good choices of x suffice to determine a, r and μ. Indeed, a e^{r} is given by a = u(1, 0). Choose an additional values for x and then use the resulting values of u at (1, x), (2, 0) and (if necessary) (2, x) to obtain expressions for a, r and μ.

9. This problem concerns the equation $\frac{\partial}{\partial t}u(t, x) = -\frac{\partial}{\partial x}q(t, x) + k(t, x)$. When particle motion is due solely to the ambient fluid moving with velocity, c, I argued that the appropriate choice for q is q = c u.. When particle motion is due solely to random motion, I argued that the right choice for q is $q = -\mu \frac{\partial}{\partial x} u$. What should q be when the particles have both random motion and motion due to the ambient motion of the fluid at velocity c? Give some justification for your choice of q.