3. Introduction to advection

This section considers equations which can be used to predict the behavior of quantities which depend on both time and space. Here is an example for the sort of analysis that lies ahead: Suppose that the wind is blowing from west to east at a constant velocity of 3 meters per second. Meanwhile, an explosion in a chemical warehouse has pumped particulate pollution into the air. These particles then fall out of the air at a constant percentage rate r. What do we need to know to predict the particle concentration east and west of the explosion as a function of time and distance from the explosion?

a) What comes in must go out

Let $u(t, x)$ be the function which gives the density (number of particles per meter) of particulate matter in the air at time t and point x along the west/east line through the warehouse. (Make the origin at the warehouse.) The following considerations lead to an equation for u: First, fix a point x and a small distance Δx. The amount of particulate matter in the region between x and $x + \Delta x$ at time t is given (approximately) by

$$u(t, x)\, \Delta x \quad . \tag{3.1}$$

(This approximation becomes more and more accurate as Δx shrinks towards zero.)

The rate of change of (3.1) with respect to time is

$$\frac{d}{dt}\left(u(t, x)\, \Delta x\right) = q(t, x) - q(t, x + \Delta x) - k(t, x)\, \Delta x \quad , \tag{3.2}$$

where

- $q(t, x)$ *is the number of particles per second which pass* x *from left to right minus the number or particles per second which pass* x *from right to left at time* t.

- $q(t, x + \Delta x)$ *is the number of particles per second which pass* $x + \Delta x$ *from left to right minus the number of particles per second which pass* $x + \Delta x$ *from right to left at time* t.
- $k(t, x)\,\Delta x$ *is the approximate number of particles which are created in the region between* x *and* $x + \Delta x$ *at time t minus the number which are destroyed in this same region at time* t. *(In the example at hand,* $k(t, x) = -r\,u(t, x)$, *but one can imagine more complicated terms here.)*

(3.3)

It is important to realize that (3.2) expresses nothing more than the tautology that the rate of change of the number of particles in the region between x and $x + \Delta x$ is given by:

- *Adding the number of particles which enter across* x *and subtracting the number which leave across* x *(the term* $q(t, x)$*).*
- *Subtracting the number which leave across* $x + \Delta x$ *and adding the number which enter across* $x + \Delta x$ *(the term* $q(t, x + \Delta x)$*).*
- *Finally, adding the number which are created and subtracting the number which are destroyed in the region between* x *and* $x + \Delta x$ *(this is given by the term* $k(t, x)\,\Delta x$*).*

If we divide both sides of (3.2) by Δx and take the limit as Δx tends to zero, we obtain

$$\frac{\partial}{\partial t}u(t, x) = -\frac{\partial}{\partial x}q(t, x) + k(t, x)\ . \tag{3.4}$$

In this regard, remember that

$$\frac{\partial}{\partial x} q(t, x) = \lim_{\Delta x \to 0} \frac{q(t, x + \Delta x) - q(t, x)}{\Delta x} \;; \tag{3.5}$$

this explains the first term on the right hand side of (3.4).

b) The form of q

Equation (3.4) is not very useful unless we can find a reasonable form for the function q(t, x) and of k(t, x). Now in the example of the explosion, we have already decided to take k(t, x) = -r u(t, x) to account for the fact that the particles are lost at a constant rate r.

In our explosion example, there is also a relatively simple form for q(t, x) if we assume that the motion of the particles is entirely due to their being pushed along by the wind. In this case, the number of particles which pass the point x from left to right at time t is given by 3u(t, x) particles per second; and the number of particles which pass from right to left is equal to zero. That is, if the wind is blowing from left to right at a speed of 3 meters per second, then the number of particles per second which pass the point x at time t is given by the density (in units of particles per meter) times the wind speed (in units of meters per second). To summarize, under the preceding assumptions, one should take

$$q(t, x) = 3u(t, x) \;. \tag{3.6}$$

Please take note of the assumption that we used to derive (3.6): The particles are moving only because of the motion of the wind. This is a reasonably valid assumption if the particles are heavy, but if the particles are very light, then one expects some random dissipative motion even without the wind blowing. (We will see subsequently how to model the dissipative case too.)

c) The advection equation

If we plug k = -r u and q = 3u into Equation (3.4), we find that the function u(t, x) is predicted to be a solution to the advection equation

$$\frac{\partial}{\partial t}u(t, x) = -3\frac{\partial}{\partial x}u(t, x) - r\, u(t, x) \;. \tag{3.7}$$

So, with the preceding assumption about the form for q(t, x), the density function u(t, x) (which is what we are interested in) is constrained in the sense that its partial derivatives in the t and the x directions are related according to (3.7). Thus, if we know the solutions to (3.7), then we know something about the form of our mysterious function u. Equation (3.7) is our first example of what is often called a partial differential equation. The use of the word ‘partial’ is in reference to the partial derivatives which appear in the equation.

FACT: *Every solution to this equation can be written as*

$$u(t, x) = e^{-rt} f(x - 3t) \tag{3.8}$$

where f(x - 3t) *means the following: Take any function* f *of one variable. Then, create the function* f(x - 3t) *of the variables* t *and* x *which is obtained by evaluating your original function* f *at the point* x - 3t.

Note that there is a huge set of solutions to (3.7), for I can choose any 1-variable function f to use in (3.8).

Here is how to prove that (3.8) solves (3.7): Simply compute the indicated partial derivatives using the Chain rule.

d) Initial conditions

There are infinitely many solutions of (3.7), one for each choice of 1-variable function f in (3.8). How does one determine the precise function f to use in (3.8)? Here is where the initial conditions enter. The term 'initial condition' signifies the value of u(t, x) as x varies but t is fixed at some predetermined time (say t = 0).

For example, suppose that the value of u at t = 0 has been determined apriori to be given as a function of x. Call this new function g(x). That is, suppose that knowledge of g(x), which is the time zero density, has been given. Then, there is one and only one solution to (3.7) with u(0, x) = g(x). This is the solution to (3.7) where the function f is precisely the function g. That is,

$$u(t, x) = e^{-rt} g(x - 3t) \tag{3.9}$$

is the only solution to (3.7) which obeys the initial condition

$$u(0, x) = g(x) \ . \tag{3.10}$$

The following summarizes this business with initial conditions:

The values of any solution u *to (3.7) at all times* t *and at all points* x *are predetermined by the time* 0 *values of* u *at all points* x.

(3.11)

In the explosion scenario above, one can imagine that a satellite photo has been taken at time 0 (a few minutes after the explosion), and that the particle density g(x) has been determined as a function of x at time 0 from the satellite photograph. Then, the values of u(t, x) at all subsequent times and all points can be predicted.

The point here is that the values $u(t_0, x)$ as x varies at fixed time $t = t_0$ completely determine the solution u(t, x) to (3.7).

e) Traveling waves

To get a feeling for what these solutions look like, consider (3.7) in the case where r = 0. As asserted above, the general solution has the form u(t, x) = f(x - 3t), where f(·) can be any function of 1-variable and u(t, x) is obtained from f by evaluating the latter at the point x - 3t. Now, what does this say about u? Among other things, it says that the value of u at a point (t, x) is the same as that of u at time 0, but not at x, rather at 3t units to the left of x. Said differently, if you were to walk to the right (increasing x) at speed 3 (so your x coordinate increases by the amount 3t after time t), then you would not see any change in the value of u. In this sense, the solution u(t, x) to the r = 0 version of (3.7) describes a concentration of particles which moves at speed 3 to the right, but otherwise maintains its shape. Likewise, the versions of (3.7) with $r \neq 0$ describe concentrations of particles which move at speed 3 to the right and either decrease ($r < 0$) or increase ($r > 0$) in time as they move.

f) Lessons

Here are some of the important lessons from this section of the handout:

- There is a tautological differential equation, (3.4), which describe the time and space dependence of the density of particles moving in a fluid. Here, q(t, x) takes into account the net flow of particles past x at time t, while k(t, x) takes into account the net number of particles created and destroyed at x at time t.

- In the case where particle motion is due to the constant velocity flow of the ambient fluid, then $q = -c\,u$, where $c > 0$ if the flow is from left to right on the x-axis, and otherwise $c < 0$. With this choice of q, (3.4) is called an advection equation.
- The advection equation in (3.7) is predictive in the sense that you can specify what you want for the function u at time $t = 0$ as a function of x and then there is a unique solution to the equation which is your specified function of x at $t = 0$.
- A solution to (3.7) resembles a traveling wave.