5. Laplace's equation

The previous section saw the derivation of the diffusion equation for a function of time and one space coordinate, x. There is an analogous diffusion equation for functions of time and two or three space coordinates which is derived by the same sort of bookkeeping considerations. Indeed, the diffusion equation for a function, u(t, x, y, z) of time and the usual three space coordinates reads:

$$\frac{\partial}{\partial t}u = \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u + k(t, x, y, z)\ . \tag{5.1}$$

The case where there are only two space coordinates, x and y, is obtained from (5.1) by deleting all references to the coordinate z.

An important solution to the k = 0 version of the R^3 version is

$$u(t, \vec{r}) = \frac{a}{t^{3/2}}\, e^{-|\vec{r}|^2/4\mu t}\ , \tag{5.2}$$

where a can be any constant. There is an analogous solution to the R^2 version; it is given by (5.2) with t replacing $t^{3/2}$ in the fraction out front. In general, there are many solutions to (5.1) no matter the form of the function k.

By the way, the combination

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{5.3}$$

of second derivatives appears so often in scientific applications that it is given a special symbol, usually Δ, although sometimes ∇^2. Note that Δu is the same as div($\vec{\nabla}$u). The combination Δ of derivatives in (5.3) is called the Laplacian.

a) The time independent case

An important equation in its own right is the diffusion equation in (1.1) for a function on R^3 or R^2 which is independent of time. In this case, the function k is also assumed to be independent of time. Thus, one is looking for u(x, y, z) (or u(x, y)) which solves the equation

$$-\Delta u = k \, . \tag{5.4}$$

This last equation is called Laplace's equation; it arises in many scientific applications, even those with no 'diffusion' involved. For example, if k is interpreted as a density of electric charge in space, then the gradient, ∇u, gives the resulting electric field in space.

b) The Green's function

In the case where u and k are functions of just one coordinate, x, then (5.4) reads

$$-u_{xx} = k \, . \tag{5.5}$$

This last equation can be solved for u by integrating twice. For example:

$$u(x) = -\int_0^x \left(\int_0^s k(\tau)d\tau\right)ds \tag{5.6}$$

However, you might be surprised to learn that there is a way of writing a solution to (5.5) which involves only a single integral (even though (5.5) involves two derivatives). In particular,

$$u(x) = -\,2^{-1}\int_{-\infty}^{\infty} |\,x-s\,|\,k(s)ds$$

(5.7)

is a solution to (5.5) when the function k is zero outside of some bounded region on R (or else |k(s)| tends to zero sufficiently rapidly as $|s| \to \infty$). It is a real test of your understanding of differentiation to take two derivatives of (5.7) and so verify that (5.5) is obeyed. By the way, note that the expression 2^{-1} |x - s| which appears in (5.7) is, for each fixed s, a bonafide function, G(x), on R. Moreover, as |x - s| = s - x where x < s and |x - s| = x - s where x > s, this function obeys the equation $G_{xx} = 0$ except at the point x = s where its second derivative does not exist.

In the R^3 and R^2 cases of (5.4), there are analogous integration formulas which provide solutions to (5.4). In particular, if $k(\vec{r})$ is a function on R^3 which is zero outside of some bounded region (or which gets small fast enough as $|\vec{r}|$ tends to infinity) then there is a unique solution, $u(\vec{r})$, to (5.4) which has the property that $|u(\vec{r})|$ also tends to zero as $|\vec{r}|$ tends to infinity. This solution is determined by the function k via a multiple integral:

$$u(\vec{r}) = \frac{1}{4\pi}\iiint \frac{k(\vec{s})}{|\,\vec{r}-\vec{s}\,|}\,ds_1 ds_2 ds_3$$

(5.8)

The expression $\frac{1}{4\pi}\frac{1}{|\,\vec{r}-\vec{s}\,|}$ which appears above is called the <u>Green's</u> <u>function</u> for the Laplacian. Note that with $\vec{s}$ fixed, this Green's function is a bonafide function of points $\vec{r}$ in R^3 except at the point $\vec{r} = \vec{s}$ where it blows up. It is left to you as an exercise to verify that this function, $G(\vec{r})$, satisfies $\Delta G = 0$ where $\vec{r} \neq \vec{s}$.

There is also a version of (5.8) for the case where u and k are functions on R^2 instead of on R^3. In this case, there is a solution, $u(\vec{r})$, to (5.4) which is given by the double integral

$$u(\vec{r}) = \frac{1}{2\pi}\iint k(\vec{s})\ln(|\vec{r}-\vec{s}|)ds_1 ds_2 \tag{5.9}$$

In this two dimensional case, $\frac{1}{2\pi}\ln(|\vec{r} - \vec{s}|)$ is called the Green's function. Again, for fixed $\vec{s}$, this is a bonafide function, $G(\vec{r})$, of $\vec{r}$ except at $\vec{r} = \vec{s}$. And, as in the 3-dimensional case, $\Delta G = 0$ where $\vec{r} \neq \vec{s}$.