Math 21a Supplement on Differential Equations from Physics

You are most probably well aware that differential equations play a central role in modern physics. So, my purpose in this supplement is not so much to sell you on differential equations but to present a few of the central equations with a brief commentary. In particular, the heat equation, Euler's equation, the wave equation, Schrodinger's equation and the Dirac equation are discussed below.

a) The heat/diffusion equation

The heat equation predicts the time-space dependence of the temperature in a homogeneous body from the spatial temperature distribution at some fixed time. To be specific, let $T(t, \mathbf{x})$ denote the temperature at time t and position $\mathbf{x} = (x, y, z)$ in a region of space. To a good approximation, this function of four variables obeys the equation

$$T_t = \mu\,(T_{xx} + T_{yy} + T_{zz}) \;, \tag{1}$$

where $\mu > 0$ is a constant (or, sometimes a function) which depends on the composition of the region.

Given $T(0, \mathbf{x})$ for all points $\mathbf{x}$ in the region, this equation is used to predict the function T at all times $t > 0$ and points $\mathbf{x}$. In this regard, note that when the region in question is not all of R^3, (1) must be supplemented with the specification of the function T at all times $t \geq 0$ on the boundary of the region. These <u>boundary</u> <u>conditions</u> are necessary for the simple reason that applying various amounts of heat to the boundary will effect the internal temperature no matter what T is at time zero. (To see this effect, take the region in question to be the interior of your body and step outside in the winter first with a jacket on and then without.)

Perhaps you recognize this equation as the diffusion equation which was introduced in Section 4 of the course-wide Handout on Differential Equations. Why does temperature obey a diffusion equation? Needless to say, the answer to this question requires an understanding of what temperature really measures, which is the average kinetic energy of the particles which make up the material. The answer also requires some understanding of how energy is transferred from particle to particle in a material (usually, but not always, collisions). Given these two facts, try to derive (1) using the ideas in Section 3a of the course-wide Handout on Differential Equations. In any event, a course on Statistical Mechanics should explain the origin of (1).

b) Euler's equation

Euler's equation is used to predict the time and space dependence of the velocity vector field of a moving, incompressible gas or fluid given the time zero spatial dependence of the velocity vector field and its time dependence at all points of the boundary of the region of interest. (Incompressibility is a term which implies that the amount of the fluid in any given volume can not be increased or decreased. Thus, the fluid can only flow through the volume. To a good approximation, water at a constant temperature is incompressible. However, most gasses are not.)

To be specific about Euler's equation, let **v**(t, **x**) denote the vector in R^3 which gives the velocity of a point in the fluid at time t and position **x**. To a good approximation, this vector function of time and space obeys the two equations

- $\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla p$,
- $\mathrm{div}(\mathbf{v}) = 0$.

(2)

Here, p(t, **x**) is the ambient pressure at time t and position **x**. Also, the symbol $(\mathbf{v}\cdot\nabla)$ a certain combination of derivatives which are to be applied to each component of **v**. To explain, first, write the components of the vector field **v** as **v** = (a, b, c) with a, b and c functions. Then, $\mathbf{v}\cdot\nabla$ signifies the following combination of derivative:

$$\mathbf{v}\cdot\nabla = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \ .$$

(3)

Thus, the components of $(\mathbf{v}\cdot\nabla)\mathbf{v}$ are

$$(a\, a_x + b\, a_y + c\, a_z,\ a\, b_x + b\, b_y + c\, b_z,\ a\, c_x + b\, c_y + c\, c_z) \ .$$

(4)

A course in Statistical Mechanics will also explain how (2) can be derived using the ideas in Section 3a of the course-wide Handout on Differential Equations. Note that there are presently many outstanding mathematical mysteries which surround Euler's equation. Indeed, here is the most well known: Can a velocity vector field at time t = 0 whose components have nice first and second derivatives evolve under Euler's into a velocity field with discontinuities (or worse pathologies)? This question may be less than abstract as real fluids exhibit shock waves (such as sonic booms) and the physics of such waves is not generally well understood.

c) The wave equation

The wave equation in its simplest form has as unknown a function p(t, **x**) of time t and position **x**. The equation reads:

$$p_{tt} - c^2\,(p_{xx} + p_{yy} + p_{zz}) = 0 \ .$$

(5)

Here, c is a constant which can be identified as the wave speed.

Basic solutions to this equation are functions of the form

$$p(t, \mathbf{x}) = A\sin(\mathbf{k}\cdot\mathbf{x} - c\,|\mathbf{k}|\,t) + B\cos(\mathbf{k}\cdot\mathbf{x} - c\,|\mathbf{k}|\,t)$$

(6)

where $\mathbf{k}$ is a fixed vector in R^3, while A and B are constants. For example, if $\mathbf{k} = (1, 0, 0)$ and B = 0, then first solution in (6) becomes the function

$$p(t, x) = A \sin(x - c\, t) \,. \tag{7}$$

Note that at any fixed x, this solution oscillates as a sinusoidal function of t with period $2\pi/c$. Meanwhile, the solution is constant along the lines where $x = c\, t$. The fact is that (7) is a wave whose amplitude is $|A|$. The time delay between successive peaks hitting the same point x for this wave is $2\pi/c$. Moreover, each peak moves along according to the rule $x = c\, t$, and so c is the speed of the wave peak.

The general solution in (6) represents a wave also. The peak amplitude is $(|A|^2 + |B|^2)^{1/2}$, while the direction of wave movement is $\mathbf{k}/|\mathbf{k}|$. Here, the time delay between successive peaks hitting the same point along a line parallel to $\mathbf{k}$ is $2\pi/(c\,|\mathbf{k}|)$. As before, the speed of the peak is again equal to c.

By the way, since (5) does not multiply p or its derivatives by anything p-dependent, the <u>superposition</u> <u>principle</u> is valid. Recall that this means that the sum of two solutions to is also a solution. This last fact can be used to prove that every solution to (5) can be written as a suitable 'sum' of the solutions in (6). Here, the word 'sum' is in quotes because usually an integral must be employed. This is to say that every solution p(t, x) to (5) can be written as a triple integral over the 3-space where $\mathbf{k} = (k_1, k_2, k_3)$ lives; here, the integrand is given by the solutions in (6) with A and B taken as functions of $\mathbf{k}$:

$$p(t, x) = \iiint (A(\mathbf{k}) \sin(\mathbf{k}\cdot\mathbf{x} - c\,|\mathbf{k}|\, t) + B(\mathbf{k}) \cos(\mathbf{k}\cdot\mathbf{x} - c\,|\mathbf{k}|\, t))\, dk_1\, dk_2\, dk_3 \,. \tag{8}$$

Here is one last remark about the wave equation: In order to use (5) to predict the $t > 0$ development of a wave, it is necessary to know both $p(0, \mathbf{x})$ and $p_t(0, \mathbf{x})$ at all points $\mathbf{x}$. This is to say that any pair of (reasonable) functions of $\mathbf{x}$ are the values of p and p_t at time zero for a solution of (5), and there is only one solution to (5) whose values at $t = 0$ and whose time derivative at $t = 0$ are the given functions.

d) Maxwell's equations

You might recall that Maxwell's equations are equations for a pair $(\mathbf{E}(t, \mathbf{x}), \mathbf{B}(t, \mathbf{x}))$ of vector fields on R^3 which depend on both time t and the spatial point $\mathbf{x}$. These equations read:

- div $\mathbf{E} = 0$,
- div $\mathbf{B} = 0$,
- $\mathbf{E}_t = \text{curl}\, \mathbf{B}$,
- $\mathbf{B}_t = -\, \text{curl}\, \mathbf{E}$.

(9)

The reason that I bring these equations up here it that the four equations in (9) imply that each of the component functions of **E** and **B** satisfy the wave equation in (5) with c = 1. To see this for **E**, take the time derivative of the third equation in (9) and use the fact that the order of taking derivatives (x, y or z first and t second, or vice-versa) is immaterial to rewrite the resulting equation as

$$\mathbf{E}_{tt} = \text{curl}\, \mathbf{B}_t \,. \tag{10}$$

Next, plug in the fourth equation in (9) to replace the time derivative of **B** and so obtain

$$\mathbf{E}_{tt} = \text{curl}(\text{curl}\, \mathbf{E}) \,. \tag{11}$$

Having got this far, write out curl(curl **B**)) and you will find it equal to

$$\text{curl}(\text{curl}\, \mathbf{E}) = \Delta\mathbf{E} - \nabla\, \text{div}(\mathbf{E}), \tag{12}$$

where $\Delta\mathbf{E} = \mathbf{E}_{xx} + \mathbf{E}_{yy} + \mathbf{E}_{zz}$. Finally, use the first equation in (9) to conclude that (11) is the same as

$$\mathbf{E}_{tt} = \mathbf{E}_{xx} + \mathbf{E}_{yy} + \mathbf{E}_{zz} \,. \tag{10}$$

A similar derivation yields the c = 1 version of the wave equation for the components of **B**.

I previously remarked that these wave equation for **E** and **B** are the c = 1 versions of (5). As the constant c gives the speed of the wave, and the wave here is that of light, you see that I have written Maxwell's equations in some special units where the speed of light is one instead of roughly 300,000 kilometers per second. If you don't want units where the speed of light is one, you must replace (9) with

- div $\mathbf{E}$ = 0,
- div $\mathbf{B}$ = 0,
- $c^{-1}\, \mathbf{E}_t = \text{curl}\, \mathbf{B}$,
- $c^{-1}\, \mathbf{B}_t = -\, \text{curl}\, \mathbf{E}$.

(11)

Using (11) instead of (9), you will obtain $\mathbf{E}_{tt} = c^2\, (\mathbf{E}_{xx} + \mathbf{E}_{yy} + \mathbf{E}_{zz})$ instead of (10), and c will be the speed of light.

e) The Schrodinger equation

The Schrodinger equation is the central equation in quantum mechanics. However, before getting to the equation, I will first digress to discuss quantum mechanics.

To begin, you must realize that 'classical' mechanics (such as Newton's laws) and quantum mechanics have different goals. have a very different where the point of 'classical' mechanics is to predict the position of a particle at times $t \geq 0$ knowing its position and velocity at time zero (and the forces on it), the goal of quantum mechanics is quite different. Roughly speaking, the goal of quantum mechanics is to predict a probability density for finding a particle at a given point at time $t \geq 0$ given certain information at time $t = 0$ and given the forces which act on the particle.

To be somewhat more precise, the quantum mechanics is concerned with a certain function, $\psi(t, \mathbf{x})$, which assigns a <u>complex</u> number to each point, $\mathbf{x}$, in space at each time t. In this regard, remember that a complex number has the form a + i b, where a and b are ordinary numbers and where $i = \sqrt{-1}$. Thus, $\psi(t, \mathbf{x})$ can be written as $a(t, \mathbf{x}) + i\, b(t, \mathbf{x})$ where a and b are functions of time and space. (If you don't already know about complex numbers, quantum mechanics will almost surely seem terribly opaque.) The function ψ is called the 'wave function'. Now, not all complex valued functions ψ are considered in quantum mechanics. In particular, there are two constraints on the allowable ψ's. Here is the first: At each time t,

$$\iiint |\psi(t, \mathbf{x})|^2 \, dx\, dy\, dz = 1 \,, \tag{12}$$

where the integration region is the whole of R^3. The other constraint will be discussed momentarily. (Remember that the absolute value of a complex number c = a + i b is the real number $(a^2 + b^2)^{1/2}$ and it is denoted by $|c|$.) The second requirement is that ψ must obey the Schrodinger equation which is presented below.

Quantum mechanics connects to the real world through the following interpretation of this wave function: The function $|\psi(t, \mathbf{x})|^2$ is postulated to give the probability density for finding, at time t, the particle at the point $\mathbf{x}$. This is to say that if V is any region in space, then the probability of the particle being in V at time t is

$$\iiint_V |\psi(t, \mathbf{x})|^2 \, dx\, dy\, dz = 1 \,. \tag{13}$$

(Thus, the constraint in (11) ensures the reasonable requirement that the probability is 1 for the particle to be somewhere.)

The dynamics of the particle enters in quantum mechanics by a requirement which specifies the way $\psi(t, \mathbf{x})$ must change with time. In particular, ψ should obey the <u>Schrodinger</u> <u>equation</u>, which relates the time derivative of ψ to ψ's second derivative in the spatial directions. For example, if there are no forces involved, the Schrodinger equation reads

$$i\, \hbar \psi_t = - \hbar^2/2m \;(\psi_{xx} + \psi_{yy} + \psi_{zz}) \;, \tag{14}$$

where i = √-1 again, $\hbar$ is a constant called 'Planck's constant', and m is the mass of the particle. (Note the resemblance to the diffusion equation in (1); you could view (14) as a diffusion equation with imaginary diffusion constant $\mu = -\hbar/2m$.)

If the particle is acted on by a force **F** which is a gradient, say **F** = ∇h where h(**x**) is a function, then the corresponding version of the Schrodinger equation reads

$$i\,\hbar\psi_t = -\,\hbar^2/2m\ (\psi_{xx} + \psi_{yy} + \psi_{zz}) + h\,\psi\ . \tag{15}$$

There are versions of Schrodinger's equation for more complicated forces too (such as magnetic forces), but I'll leave you to learn about these in your courses on quantum mechanics.

f) The Dirac equation

Like Newton's equation, the Schrodinger equation does not take special relativity into account. In particular, (14) does not give correct predictions. Although it is extremely accurate for electrons with speeds much less than that of light, its predictions are not accurate for speeds that are a significant fractions of the speed of light. The problem of reconciling the quantum mechanics of the electron with relativity was solved by Paul Dirac in the manner which I will now describe.

To begin, Dirac postulated that the wave function in (14) should be replaced by a 4 by 1 matrix of functions, Ψ(t, **x**), whose four entries are each complex valued functions of space and time. Thus,

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \tag{16}$$

where each of $\psi_{1,2,3,4}$ is assigns to each pair (t, **x**) of time t and space point **x** a complex number. Then, the Schrodinger equation in (14) is replaced by the equation

$$i\,\hbar\Psi_t = -\,i\,\hbar\,c\,(\alpha_1\,\Psi_x + \alpha_2\,\Psi_y + \alpha_3\,\Psi_z) + m\,c^2\,\beta\,\Psi\ , \tag{17}$$

where c is the speed of light, $\hbar$ is Planck's constant, m is the electron's mass, and α_1, α_2, α_3 and β are certain 4×4 matrices with constant (though complex valued) entries. These matrices must satisfy the following matrix multiplication rules:

- $\beta\beta = \alpha_1\alpha_1 = \alpha_2\alpha_2 = \alpha_3\alpha_3 = I$,
- $\beta\alpha_1 + \alpha_1\beta = \beta\alpha_2 + \alpha_2\beta = \beta\alpha_3 + \alpha_3\beta = 0$,
- $\alpha_1\alpha_2 + \alpha_2\alpha_1 = \alpha_2\alpha_3 + \alpha_3\alpha_2 = \alpha_3\alpha_1 + \alpha_1\alpha_3 = 0$.

(18)

Here, I signifies the 4×4 matrix with only diagonal entries, and all of these equal to 1.

It turns out that any choice of four matrices which obey (18) will give the correct answer for high speed electrons when used in (17). One solution takes β to be diagonal with entries 1, 1, -1, -1 as you move down the diagonal from top left to bottom right. Meanwhile, α_1 and α_2 are 'anti-diagonal', which is to say that their only non-zero entries are on the line running from lower left to upper right. For α_1, all entries on this line are +1, while for α_2, the entries are i, -i, i, -i moving from lower left to upper right. I'll leave it as a challenge for you to determine α_3. (Hint: Its entries are ±1 and it has zero's on the diagonal and the anti-diagonal.)

Here is a challenge: Prove using (17) and (18) that the entries of Ψ obey the following generalization of the wave equation:

$$\Psi_{tt} = c^2 (\Psi_{xx} + \Psi_{yy} + \Psi_{zz}) - m^2 c^4 \hbar^{-2} \Psi \qquad (19)$$

(Hint: Start by taking the time derivative of both sides of (17).)

By the way, the physical interpretation of Ψ is more complicated then you might guess. Indeed, it is not the case that $|\Psi|^2 = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$ gives the probability density for finding, at time t, an electron at position **x**. In fact, to get a consistent connection between Ψ and the real world, Dirac was forced to postulate the existence of a new fundamental particle, an 'anti-particle' for the electron. (The latter is now called the positron.) The connection between Ψ and the real world involves both the electron and the positron. Remarkably enough, positrons were observed in nature just a few years after Dirac was lead to postulate their existence.