

## Math 21a Supplement on Planetary Motion

Suppose that an object (planet, asteroid, whatever) travels through space under the gravitational influence of a star. Newton's laws allow one to describe trajectory of the planet. This supplement describes the situation.

### a) Newton's Laws

There is a natural coordinate system to describe the motion. This is the so called center of mass coordinate system. (See the final section, f.) In this coordinate system, the position vector  $\mathbf{r}(t)$  for the object at time  $t$  (the vector from the origin to the planet's position) evolves in time according to a differential equation of the following form:

$$\frac{d^2}{dt^2} \mathbf{r} = -\kappa \mathbf{r}/r^3 . \tag{1}$$

Here,  $\kappa$  is a positive constant which is determined by the masses of the object and the star. Also,  $r = |\mathbf{r}|$  is the distance of the object from the origin.

To simplify the typing and notation, allow me to introduce the following shorthand:

- $\frac{d}{dt} \mathbf{r} \equiv \mathbf{r}' .$
- $\frac{d^2}{dt^2} \mathbf{r} \equiv \mathbf{r}'' .$

(2)

For example, Equation (1) reads

$$\mathbf{r}'' = -\kappa \mathbf{r}/r^3 . \tag{3}$$

### b) The energy

The standard strategy for solving (3) is to find so called "constants of the motion". These are quantities which are constant along each trajectory. The energy,

$$E = |\mathbf{r}'|^2/2 - \kappa/r . \tag{4}$$

is one such constant. To show that  $E$  is constant, consider its time derivative under the assumption that  $\mathbf{r}(t)$  obeys (3). One finds that

$$E' = \mathbf{r}' \cdot \mathbf{r}'' + \kappa \mathbf{r}' \cdot \mathbf{r}/r^3 . \tag{5}$$

The fact that the latter is zero follows by taking the dot product of both sides of (3) with the vector  $\mathbf{r}'$ .

The derivation of (5) uses the fact that  $r = (\mathbf{r} \cdot \mathbf{r})^{1/2}$  and that the time derivative of a dot product obeys

$$(\mathbf{A} \cdot \mathbf{B})' = \mathbf{A}' \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{B}' \tag{6}$$

when  $\mathbf{A}$  and  $\mathbf{B}$  are both vector valued functions of time. (Equation (6) can be verified by writing out  $\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$  and then differentiating.)

You should think about what it means if the energy  $E$  is constant along a trajectory. For example, suppose  $E$  is negative. Can the trajectory get arbitrarily far from the origin? No, in fact, according to (4),

$$\kappa/r = |\mathbf{r}'|^2/2 - E \geq -E . \quad (7)$$

If  $(-E) > 0$ , then multiplying both sides above by  $r/(-E)$  doesn't change the direction of the inequality and produces

$$\kappa/(-E) \geq r . \quad (8)$$

Thus, if  $E$  is negative, the orbit is necessarily bounded.

What about when  $E$  is zero or positive? As we shall see, this case allows orbits which are unbounded.

### c) The angular momentum

Another constant of the motion is the angular momentum,

$$\mathbf{L} = \mathbf{r} \times \mathbf{r}' . \quad (9)$$

To see that this is independent of  $t$  when  $\mathbf{r}(t)$  obeys (3), simply differentiate to find that

$$\mathbf{L}' = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = 0 + \mathbf{r} \times (-\kappa \mathbf{r}/r^3) = 0 + 0 = 0 . \quad (10)$$

The derivation here uses the fact that the time derivative of the cross product of two time dependent vectors obeys

$$(\mathbf{A} \times \mathbf{B})' = \mathbf{A}' \times \mathbf{B} + \mathbf{A} \times \mathbf{B}' , \quad (11)$$

as can be verified by writing out  $\mathbf{A} \times \mathbf{B} = (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$  and then differentiating. Also, the derivation of (10) uses the fact that  $\mathbf{V} \times \mathbf{V} = 0$  for any vector  $\mathbf{V}$ .

By the way, the fact that  $\mathbf{L}$  is constant means that each  $\mathbf{r}(t)$  is at  $90^\circ$  to  $\mathbf{L}$ . Thus, the whole trajectory lies in the plane through the origin whose normal vector is  $\mathbf{L}$ . Of course, this assumes that  $\mathbf{L} \neq \mathbf{0}$ . If  $\mathbf{L} = 0$ , then this means that  $\mathbf{r}'$  is proportional to  $\mathbf{r}$  at each point. That is,  $\mathbf{r}' = \alpha(t) \cdot \mathbf{r}$ , where  $\alpha(t)$  is some function of  $t$ . This means that the whole trajectory lies upon a line. Indeed, this last equation implies that the unit vector  $\mathbf{r}/r$  is constant (when  $r \neq 0$ ), as one can see by differentiating.

### d) Reducing to the plane

To find the trajectory when  $\mathbf{L} \neq 0$ , it pays at this point to focus attention to the plane on which  $\mathbf{r}$  lies. Now one plane is pretty much the same as any other, and with this understood, one can set up a coordinate system on the plane through the origin perpendicular to  $\mathbf{L}$  plane with coordinates, say  $(u, v)$ . Furthermore, it is convenient to introduce *polar coordinates* in this plane, by writing  $u = r \cos\theta$  and  $v = r \sin\theta$ . With this understood, then  $r$  and  $\theta$  are functions of  $t$  along the trajectory. That is,

$$\mathbf{r} = (r \cos\theta, r \sin\theta) \quad \text{and} \quad \mathbf{r}' = (r' \cos\theta - r \theta' \sin\theta, r' \sin\theta + r \theta' \cos\theta) . \quad (12)$$

With respect to these coordinates, we find that  $|\mathbf{L}| = |\lambda|$ , where  $\lambda$  is a constant given by

$$\lambda = r^2 \theta' . \quad (13)$$

Meanwhile,

$$E = (r'^2 + r^2 \theta'^2)/2 - \kappa/r . \quad (14)$$

Now, Equations (13) and (14) can be used to solve for  $r'$  and  $\theta'$  in terms of  $r$  and the constants  $E$  and  $\lambda$ . Indeed, (13) directly gives  $\theta' = \lambda/r^2$  and (14) with this last identity for  $\theta'$  implies that

$$r' = \pm(2E - \lambda^2/r^2 + 2\kappa/r)^{1/2} . \quad (15)$$

The preceding equation is a differential equation for  $r(t)$  which can be integrated to give

$$\int (2E - \lambda^2/r^2 + 2\kappa/r)^{-1/2} \cdot dr = \pm t \quad (16)$$

### e) The trajectory

The integral in (16) can be evaluated. However, it is also instructive to consider the trajectory as tracing out a path in the  $(u, v)$  plane. This path is described by giving the distance  $r$  as a function of the angle  $\theta$ . In this case, (15) and the equality  $\theta' = \lambda/r^2$  imply the relation

$$\frac{dr}{d\theta} = \pm (2E - \lambda^2/r^2 + 2\kappa/r)^{1/2} r^2/\lambda , \quad (17)$$

which can be shown to be the equation of a conic section with the origin as one of the focal points. In particular, the  $E < 0$  case is an ellipse, the  $E = 0$  case is a parabola and the  $E > 0$  case is one branch of a hyperbola. (The eccentricity here is  $(1 + 2E\lambda^2/\kappa^2)^{1/2}$ .)

By the way, to see how a conic section arises from (17), make the substitution  $w = 1/r - \kappa/\lambda^2$ . Then, by virtue of (17),  $w$  obeys

$$-\frac{dw}{d\theta} = \pm(2E + \kappa^2/\lambda^4 - w^2)^{1/2} , \quad (18)$$

which can be solved (using integration formulas from Math 1b) by the further substitution  $w = (2E + \kappa^2/\lambda^4)^{1/2} \cdot \cos(\theta - \theta_0)$  where  $\theta_0$  is a constant of integration. Putting this back in terms of  $r$  gives the equation for the conic section,

$$r = (\lambda/\kappa)^2 / (1 + (2E\lambda^2/\kappa^2 + 1)^{1/2} \cos(\theta - \theta_0)) . \quad (19)$$

**f) The center of mass**

As remarked at the outset, the vector  $\mathbf{r}$  in (1) is the position vector for the object where the origin is at the center of mass of the system. Note that this center of mass point is not at the center of sun. Indeed, just as the center of the planet moves due to the gravitational force from the sun, so the center of the sun moves with time due to the gravitational force from the planet. Thus, the center of the sun is not a point which is fixed in time. Anyway, in an ideal world run by Newton's laws, here is how to find the center of mass point,  $\mathbf{c}$ : First, choose any fixed coordinate system, and let  $\mathbf{x}$  denote the position vector of the planet and  $\mathbf{X}$  that of the sun. Meanwhile, let  $m$  denote the mass of the planet and  $M$  that of the sun. Then, in this chosen coordinate system,

$$\mathbf{c} = m \mathbf{x} + M \mathbf{X} . \tag{20}$$

The center of mass coordinate system is then chosen so that  $\mathbf{c} = 0$ . Newton's laws insure that  $\mathbf{c}$  remains zero for all time as long as the system is not acted upon by additional forces.

Note that in the center of mass coordinate system,  $|\mathbf{X}| = m |\mathbf{x}|/M$ , so that when the sun is much more massive than the planet, then the center of mass is very close to (but not exactly) the center of the sun. In particular, this distance is relatively small when the planet in question has small mass; such is the case when the planet is Earth. However, the difference between these two points is not so small when the planet is large like Jupiter or Saturn. In particular, if you graduate from Harvard and go on to work for NASA designing probes to the outer planets, please use the correct origin for your coordinates, or else your multi-million dollar probe will miss its target.

By the way, the constant  $\kappa$  in (1) and (3) is given in terms of  $m$ ,  $M$  and the gravitational constant  $G$  by the equation  $\kappa = GM/(1 + m/M)^2$ .