Math 21a Supplement 1 on Work and Energy

Newton's law asserts that the position vector $\mathbf{r}(t)$ of a particle of mass m under the influence of a force $\mathbf{F}$ obeys the equation

$$m\,\mathbf{r}'' = \mathbf{F}\,. \tag{1}$$

a) Work

Suppose that the components of the force vector $\mathbf{F}$ do not depend on time or the position of the particle. Thus, $\mathbf{F}$ has components (a, b, c) which are numbers, not functions. (For example, $\mathbf{F} = (1, 2, 3)$.) Then, the <u>work</u> done in moving the particle from position $\mathbf{r}_0$ to position $\mathbf{r}_1$ is (by definition)

$$W \equiv \mathbf{F}\bullet(\mathbf{r}_1 - \mathbf{r}_0). \tag{2}$$

Note that this notion of work can be generalized to apply to any force vector $\mathbf{F}$, constant or not; but we are not ready at this point in the course for the generalization.

b) Energy

The <u>energy</u> of a particle at position $\mathbf{r}$ with velocity vector $\mathbf{r}'$ is the function

$$e \equiv 2^{-1}\, m\, |\mathbf{r}'|^2 - \mathbf{F}\bullet\mathbf{r}\,. \tag{3}$$

Note that Newton's law (Equation (1)) implies that the energy function does not change as time evolves. Indeed, we can differentiate (3) to find that

$$e' = m\,\mathbf{r}''\bullet\mathbf{r}' - \mathbf{F}\bullet\mathbf{r}' = \mathbf{F}\bullet\mathbf{r}' - \mathbf{F}\bullet\mathbf{r}' = 0\,, \tag{4}$$

where the second equality comes by substituting for $m\,\mathbf{r}''$ using Equation (1).

Note that the constant force vector is not the only kind of force for which energy can be defined and for which the energy is independent of time along the trajectory. Consider, for example the following case: Let f(r) be a function of the distance, $r \equiv |\mathbf{r}|$, of the particle from the origin, and consider the force $\mathbf{F} = f(r)\,\mathbf{r}/r$. For example, if a force is gravitational and due to a mass M at the origin, then $f(r) = -\,G\,m\,M\,r^{-2}$, where G is the <u>Gravitational</u> <u>constant</u>. In any event, when $\mathbf{F} = f(r)\,\mathbf{r}/r$, Newton's law reads

$$m\,\mathbf{r}'' = f(r)\,\mathbf{r}/r\,. \tag{5}$$

In this case, the energy is defined to be

$$e \equiv 2^{-1} m |\mathbf{r}'|^2 - V(r) , \tag{6}$$

where V(r) is an anti-derivative of f(r). That is, $\frac{dV}{dr} = f$. For the case where $f = G\, m\, M\, r^{-2}$, this function V(r) (called the <u>potential</u>) can be taken to be $V(r) = G\, M\, m\, r^{-1}$.

In any case, with $\mathbf{F} = f(r)\, \mathbf{r}/r$, the energy is also constant along a particle's trajectory as can be seen by first differentiating and using the Chain rule to find that

$$e' = m\, \mathbf{r}''\bullet\mathbf{r}' - f(r)\, r' . \tag{7}$$

One then employs the formula $r' = \mathbf{r}'\bullet\mathbf{r}/r$ (which is also an application of the Chain rule together with the fact that $r = (\mathbf{r}\bullet\mathbf{r})^{1/2}$). This allows (7) to be written as

$$e' = m\, \mathbf{r}''\bullet \mathbf{r}' - f(r)\, \mathbf{r}'\bullet\mathbf{r}/r . \tag{8}$$

Finally, use Equation (5) to replace $m\, \mathbf{r}''$ in this last equation by $f(r)\, \mathbf{r}/r$ to see that e' vanishes.