

Math 21a Supplement 2 on Work and Energy

As mentioned at the end of Section 5.1 in the text, the work done in moving a particle along a path γ in \mathbb{R}^3 when a force vector \mathbf{F} acts is defined to be

$$w(\gamma) \equiv \int_{\gamma} \mathbf{F} \cdot d\mathbf{x} \tag{1}$$

This is to say that term ‘work done in moving along γ ’ in physics has a specific meaning, the latter being the value of (1). (Presumably, this mathematical definition of work meshes well with our intuitive idea of work done.)

Let me remind you (from the definition in the text) that the shorthand on the right side of (1) has the following meaning: Parameterize the path γ by choosing an interval, $[a, b]$, on the line, and a vector valued function $\mathbf{x}(t)$ for $a \leq t \leq b$ which traces out the path γ . With this done, then the right side of (1) is computed by the ordinary integral

$$w(\gamma) = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt . \tag{2}$$

Now, suppose that the path γ is the path that the particle would have travelled were its trajectory γ given by Newton’s law; thus $\mathbf{x}(t)$ obeys

$$m \mathbf{x}'' = \mathbf{F} . \tag{2}$$

In this case, if we substitute $m \mathbf{x}''$ for \mathbf{F} in (2) we see that

$$w(\gamma) = m \int_a^b \mathbf{x}'' \cdot \mathbf{x}' dt , \tag{3}$$

which is nothing more than

$$w(\gamma) = 2^{-1} m \int_a^b \frac{d}{dt} |\mathbf{x}'|^2 dt = (2^{-1} m |\mathbf{x}'|^2) \Big|_{t=b} - (2^{-1} m |\mathbf{x}'|^2) \Big|_{t=a} . \tag{4}$$

(The final equality in (4) is just the Fundamental Theorem of Calculus.) In any event, (4) demonstrates that the work done by a particle moving according to Newton’s law is given by the difference between the squared norms of the velocities times the mass divided by 2.

This last formula is not unrelated to the notion of conservation of energy. In this regard, suppose that $\mathbf{F} = \nabla V$, that is the force is the gradient of a potential. (For example, the gravitational force from a point particle of mass M at the origin is the gradient of $V = G m M r^{-1}$, where $r = |\mathbf{x}|$ is

the distance to the origin.) Anyway, if $\mathbf{F} = \nabla V$, the energy of a particle moving on the trajectory $\mathbf{x}(t)$ at time t is defined to be

$$e \equiv \frac{1}{2} m |\mathbf{x}'|^2 - V(\mathbf{x}). \quad (5)$$

It then turns out that $e' = 0$ when the particle moves according to Newton's law $m \mathbf{x}'' = \nabla V$ as can be seen by differentiating (5). In this regard, use the Chain rule to write

$$V(\mathbf{x}(t))' = \nabla V \cdot \mathbf{x}', \quad (6)$$

while writing $\frac{1}{2} m (|\mathbf{x}'|^2)' = m \mathbf{x}'' \cdot \mathbf{x}'$ and then substituting ∇V for $m \mathbf{x}''$ to see that the time derivative of the first term on the right side in (5) just cancels that in the second.

The relation of all of this to (4) is that when e is constant, then the difference on the far right side of (4) is the same as

$$w(\gamma) = V(\mathbf{x}(t_b)) - V(\mathbf{x}(t_a)). \quad (7)$$

Thus, the work done when a particle moves by Newton's law against a force which is a gradient is given by the difference between the values of the potential function at the end and beginning points of the motion. (We will see in Section 5.2 of the text that this last conclusion is true even if the particle ignores Newton's law.)