Math 21a Hourly 2 Answers
(Fall 2001)

1. (a) False (example: $f(x,y) = y \exp(x)$)
   (b) True (at a stationary point, $u_{xx}=1$, so no stationary point can be a local maximum)
   (c) False ($L(1,1,1)$ and $f(1,1,1)$ are not equal)
   (d) True (by the chain rule any stationary point of f is also stationary for h)
   (e) False (the integrals are over different regions)

2. The condition $\nabla f = 0$ is equivalent to the two equations

$$6x^2 - 6y = 0 \quad \text{and} \quad -6x + 6y = 0.$$

Thus, $x = y = x^2$. There are two solutions, $(x, y) = (0, 0)$ and $(x, y) = (1, 1)$.
The second derivative matrix at a general point (x, y) is

$$\begin{pmatrix} 12x & -6 \\ -6 & 6 \end{pmatrix}$$

This matrix has determinant $72x - 36$ which is negative at $(0, 0)$ and positive at $(1, 1)$. Thus, $(0, 0)$ is a saddle. Meanwhile, $(1, 1)$ is a minimu since $f_{xx} > 0$ at $(1, 1)$.

Remarks on grading: This is a fairly standard problem, so the grading was strict. Common mistakes: a) Getting extra stationary points (this is not a Lagrange multiplier problem: you shouldn't have any extra suspects). b) Wrong formulas for the determinant (which I think is a major mistake). c) Numerical mistakes.

3. Use Lagrange multipliers and set $\nabla N = \lambda \nabla g$. Most people wrote the partial derivatives in a form such as $N_x = 0.4\, x^{-0.6} y^{0.3} z^{0.2}$ but this is equivalent to $0.4\, N/x$, a form which makes the algebra easier to follow. (An even cleaner formulation is to maximize $\ln N = 0.4 \ln x + 0.3 \ln y + 0.2 \ln z$). In any event,

$$\nabla N = (0.4\, N/x\, ,\ 0.3\, N/y\, ,\ 0.2\, N/z) \quad \text{and} \quad \nabla g = (2, 1, 1)\, ,$$

So $0.4\, N/x = 2\lambda$ and $0.2\, N/x = \lambda$ while $0.3\, N/y = \lambda$ and $0.2\, N/z = \lambda$. Equating these three expressions that all equal lambda, you will find that $0.3\, N/y = 0.2\, N/x$ which means that $y = 3x/2$. Meanwhile, $0.2\, N/z = 0.2\, N/x$ so $z = x$. Now, use the constraint: $2x + y + z = 9$ to conclude that $2x + 3x/2 + x = 9x/2 = 9$ and so $x = 2$. Thus, the maximum occurs for $x = 2$, $y = 3$, $z = 2$. At that point $N = 1728^{0.1}$.

Notes on grading: More than half the answers were perfect. Finding (2,3,2) but neglecting to compute N(2,3,2) cost one point. Failing to solve the equations correctly cost 1 - 3 points. Miscalculating the partial derivatives in ∇N was a major error, but
some credit was given for stating the Lagrange multiplier approach correctly.

4. Since the region is bounded the maximum and minimum of f exist. Thus the problem is
divided into 3 parts: (1) Find the stationary points of f in the interior of the region. (2) Find the stationary points on the boundary using Lagrange multipliers. (3) Compare the values of f at the above points to dtermine which is the maximum and which is the minimum.

To implement Step 1, note that $\nabla f = (4x + 3y, 4y + 3x, 2z)$ which is zero only at the origin. People who did not do Step 1 lost 2 points and people who 'proved' that the origin is a local minimum using an incorrect version of the second derivative test in 3 dimensions lost 1 point.

The Lagrange multiplier method finds for Step 2 the following possible points for maxima or minima on the boundary: $(0, 0, \pm 1)$, $(\pm 1/2, \pm 1/2, 0)$. Those who got only the first pair of points lost 3 points. Other mistakes cost 1 or two points.

To implement Step 3, plug in to find the value of f at the various points. The minimum is at (0, 0, 0) where $f = 0$ and the maximum, $f = 7/4$, is achieved at two points, (1/2, 1/2, 0) and (-1/2, -1/2, 0)

5. a) The solution is the double integral

$$\int_0^{\pi/2} \left(\int_0^{\cos^2(y)} dx \right) dy$$

Grading: Two points were awarded for the correct answer but only one point if one of the limits was slightly wrong. Many people had y going from 0 to 1 or the x limits from $\cos^2(y)$ to 1. Many had the integral was over a rectangular region or a single integral that would still give the right answer.

b) The solution is $\pi/4$. Grading: Eight points were awarded for the correct answer. If the double integral that given for Part a was evaluated correctly, the full eight points wee given even if it was the wrong integral. Careless arithmetic cost one or two points; two here if the answer was riddled with such errors. Other scores were awarded based on the grader's ability to decipher a correct stab at solving the integral.

6. a) The best way to approach uses the linear approximation for f based at (1, 1, 1) because both f and ∇f at (1, 1, 1) are directly computable. Indeed, as f at (1, 1, 1) is –2. Meanwhile, $\nabla f = e^{(x+y-z-1)} \cos(e^{(x+y-z-1)}$ (1, 1, -1) – (0, 0, 2) which is (1, 1, -3) at the point (1, 1, 1). This understood, the linear approximation for f that is based at (1, 1, 1) is the function L(x, y, z) = f(1, 1, 1) + $\nabla f|_{(1, 1, 1)}\cdot$(x – 1, y – 1, z – 1); this being equal to -1 + x + y – 3z. Taken at (1.01, 1.01, 1.01), the linear approximation has value –2.01. (Although you weren't expected to be able to do this, this approximation is within .001 of the true answer.)

Grading: This part of the problem was worth 6 points. Points were knocked off if the gradient was computed incorrectly or if another strategy for estimation was incorrectly applied. Additionally, points were knocked off when the grader felt that the justification was muddled.

b) The function $\sin(e^{(x+y-z-1)}$ - 1) is constant only on those planes where x + y – z is constant. Thus, the answer is: Any plane given by x + y – z = constant.

Grading: Two points were awarded if some x + y – z = constant plane was indicated. No points were awarded for the claim that there were no such planes.