

1) ___ 2) ___ 3) ___ 4) ___ 5) ___ 6) ___ 7) ___ :

Total _____

Name: _____Circle the name of your section TA:

Arinkin(10) Arinkin(12) Bamberg Cornut Kaplan Karu Knill Libine Liu Taubes Williams

Instructions:

- Print your name in the line above and circle the name of your section TF.
- Answer each of the questions below in the space provided. If more space is needed, use the back of the facing page or on the extra blank pages at the end of this booklet. Please direct the grader to the extra pages used.
- Please give justification for answers if you are not told otherwise.
- Please write neatly. Answers which are deemed illegible by the grader will not receive credit.
- No calculators, computers or other electronic aids are allowed; nor are you allowed to refer to any written notes or source material; nor are you allowed to communicate with other students. Use only your brain and a pencil.
- You have 2 hours to complete your work.
- Note that vectors are indicated below by bold face type.

In agreeing to take this exam, you are implicitly agreeing to act with fairness and honesty.

1. Write T next to those statements below that are true and F next to those that are false. There is no need to justify your answers. Note that all functions that are referred to below should be assumed to be differentiable, with differentiable partial derivatives.

- a) All functions $u(x, y)$ that obey $u_x = u$ at all points obey $u_y = 0$ at all points.
 b) Any function $u(x, y)$ that obeys the differential equation $u_{xx} + u_x - u_y = 1$ has no local maxima.
 c) The best linear approximation based at $(1, 1, 1)$ to the function $f(x, y, z) = x^3 + y^3 + z^3$ is the function $L(x, y, z) = 3x + 3y + 3z$.
 d) If $f(x, y, z)$ is any function on \mathbf{R}^3 and g is any function on \mathbf{R} then the function $h(x, y, z) = g(f(x, y, z))$ must have at least as many stationary points as does f .
 e) If $f(x, y)$ is any function of two variables, then $\int_0^1 \left(\int_x^1 f(x, y) dy \right) dx = \int_0^1 \left(\int_y^1 f(x, y) dx \right) dy$.

Answer:

- a) False, the function $u(x, y) = e^x y$ for example satisfies $u_x = u$ and $u_y = e^x$.
 b) True. At a local maximum, $u_x = u_y = 0$ and $\det(H) \geq 0$, where H is the Hessian but $H_{11} = u_{xx} = 1$ is inconsistent with a local maximum.
 c) False. The linear approximation at $(1, 1, 1)$ is $f(1, 1, 1) + \nabla f(1, 1, 1) \cdot (x - 1, y - 1, z - 1) = 3 + 3(x - 1) + 3(y - 1) + 3(z - 1) = 3x + 3y + 3z - 6$.
 d) True. The chain rule tells $\nabla h(x, y, z) = \nabla g(f(x, y, z)) \cdot f'(x, y, z)$. So if (x, y, z) is a stationary point for f , then it is also a stationary point for h .
 e) False. Drawing the triangular region shows

$$\int_0^1 \left(\int_x^1 f(x, y) dy \right) dx = \int_0^1 \left(\int_0^y f(x, y) dx \right) dy .$$

2. Find all stationary points of $f(x, y) = 2x^3 - 6xy + 3y^2$ and classify each as being either a local maximum, local minimum or saddle point.

Answer. $\nabla f(x, y) = (f_x(x, y), f_y(x, y)) = (6x^2 - 6y, -6x + 6y)$ vanishes for $x = y, x^2 - x = 0$ which means that $(0, 0), (1, 1)$ are the stationary points. The Hessian is $H(x, y) = \begin{bmatrix} 12x & -6 \\ -6 & 6 \end{bmatrix}$. We have $\det H(0, 0) = -36$ which implies that $(0, 0)$ is a saddle point. Furthermore $\det H(1, 1) = 36$ and $H_{11}(1, 1) = 12$ so that $(1, 1)$ is a local minimum.

Remark on the grading: This is a fairly standard problem, so the grading was strict. Common mistakes:
 a) Getting extra stationary points (this is not a Lagrange multiplier problem). b) Wrong formulas for the determinant. c) Numerical mistakes.

3. Find the maximum of $N(x, y, z) = x^{0.4}y^{0.3}z^{0.2}$ subject to the constraints $x \geq 0, y \geq 0, z \geq 0$ and $G(x, y, z) = 2x + y + z = 9$. In this regard, note that the region involved is bounded and $N = 0$ on $x = 0$ or $y = 0$ or $z = 0$.

Answer We have $\nabla N(x, y, z) = (0.4N/x, 0.3N/y, 0.2N/z)$, $\nabla G = (2, 1, 1)$. The Lagrange equations are $N = 0.4\lambda 2x$, $N = 0.3\lambda y$, $N = 0.2\lambda z$, $2x + y + z = 9$. The first three equations mean that $y = 3x/2, z = x$, so that $2x + 3x/2 + x = 9x/2 = 9$ and $x = 2$. The maximum is at $(2, 3, 2)$. The value of N there is $N = 1728^{0.1}$.

Remark. The problem could be simplified also by minimizing $0.4\log(x) + 0.3\log(y) + 0.2\log(z)$.

Remark on grading: More than half then answers were perfect. Finding $(2, 3, 2)$ but neglecting to compute $N(2, 3, 2)$ costed one point. Failing to solve the equations correctly costed 1 to 3 points. Miscalculating the partial derivative $\sin \nabla N$ was a major error but some credit was given for stating the Lagrange multiplier approach correctly.

4. Let $f(x, y, z) = 2x^2 + 3xy + 2y^2 + z^2$ and let R denote the region in \mathbf{R}^3 , where $2x^2 + 2y^2 + z^2 \leq 1$. Find the maximum and minimum values of f on the region R and list all points, where said maximum and minimum values are achieved.

Answer. Since the region is bounded, the maximum and minimum of f exists.

I) Critical points in the interior. $\nabla f(x, y, z) = (4x + 3y, 4y + 3x, 2z)$ vanishes at $(0, 0, 0)$.

1) $(0, 0, 0)$, $f(0, 0, 0) = 0$.

II) The Lagrange equations are $(4x + 3y, 4y + 3x, 2z) = \lambda(4x, 4y, 2z)$. The critical points on the boundary are 2) $(0, 0, 1)$, $f(0, 0, 1) = 1$, 3) $(0, 0, -1)$, $f(0, 0, -1) = 1$, 4) $(1/2, 1/2, 0)$, $f(1/2, 1/2, 0) = 7/4$, 5) $(-1/2, 1/2, 0)$, $f(-1/2, 1/2, 0) = 1/4$, 6) $(1/2, -1/2, 0)$, $f(1/2, -1/2, 0) = 1/4$, 7) $(-1/2, -1/2, 0)$, $f(-1/2, -1/2, 0) = 7/4$.

III) Compare the values: $(0, 0, 0)$ is the minimum, $(1/2, 1/2, 0)$ and $(-1/2, -1/2, 0)$ are the maxima.

5. a) Express $\int_0^1 \arccos(\sqrt{x}) dx$ as an integral over a non-rectangular region in the $x - y$ plane.
b) Evaluate the double integral you just wrote with the help of the identity $\cos^2(t) = (1 + \cos(2t))/2$.

Answer.

a) The integral $\int_0^1 \arccos(\sqrt{x}) dx$ can be written as a double integral $\int_0^1 \int_0^{\arccos(\sqrt{x})} dy dx$. Turning around the order of integration gives $\int_0^{\pi/2} \int_0^{\cos^2(y)} dx dy$.

b) $\int_0^{\pi/2} \int_0^{\cos^2(y)} dx dy = \int_0^{\pi/2} \cos^2(y) dy = \int_0^{\pi/2} (1 + \cos(2t))/2 dt = \pi/4$.

Remark to the grading: 8 points were awarded for the correct answer. If the double integral for part a) was correct, 8 points were given even if it was the wrong integral. Careless arithmetic costed one or two points.

6. Let $f(x, y, z) = \sin(e^{x+y-z-1} - 1) - 2z$.
a) Which number between -10 and 10 with two or less digits to the right of the decimal point is plausibly closest to the value of the function f at $(1.01, 1.01, 1.01)$? To ensure full credit, please provide a sentence to explain your reasoning.
b) Specify those planes in \mathbf{R}^3 on which $\sin(e^{x+y-z-1} - 1)$ has the same value at all points. If you decide there are no such planes, explain why.

Answer. a) The linear approximation of $f(x, y, z)$ at $(1, 1, 1)$ is $f(1, 1, 1) + \nabla f(1, 1, 1) \cdot (x - 1, y - 1, z - 1)$. We have $\nabla f(x, y, z) = \cos(e^{x+y-z-1} - 1)e^{x+y-z-1}(1, 1, -1) - (0, 0, 2) = (1, 1, -3)$. $f(1.01, 1.01, 1.01)$ is approximated by $f(1, 1, 1) + (1, 1, -3) \cdot (0.01, 0.01, 0.01) = -2.01$. (Additional information which was not asked: this approximation is within 0.001 of the true answer).

b) $\sin(e^{x+y-z-1} - 1)$ is constant if $x + y + z$ is constant. Therefore, the function is constant on planes $x + y - z = c$.

Remark to the grading: a) 6 points were given. Points were knocked off if the gradient was computed incorrectly or if an other strategy for an incorrectly applied estimate.

b) 4 points. two points were awarded if some $x + y - z = \text{constant}$ plane was indicated. Two points for a correct justification. No points for the claim that there were no such planes.