

ANSWERS:

- $x - (2 - 2y + y^2)^{1/4} = 0$
 - $(x^2 + 4)^{1/2} - 2y = 0.$
 - $x^2/16 + y^2/9 = 1.$
- $x = 1.$
 - $27x + 4y = -73.$
 - $x + 3y = -1.$
- $e^2 + 1.$
- $(t \cos(t^3), t \sin(t^3)).$
 - $x \sin((x^2 + y^2)^{3/2}) - y \cos((x^2 + y^2)^{3/2}) = 0.$
- $\mathbf{B} = 1/3 \mathbf{A} + 1/3 (2, 4, -5).$
 - $\mathbf{B} = 11/25 \mathbf{A} + 1/25 (92, 69, 25).$
 - $\mathbf{B} = -1/3 \mathbf{A} + 1/3 (11, 8, 7).$
- Only in Case c) are \mathbf{v} and \mathbf{w} perpendicular.
- $2/3.$
 - $(13)^{1/2}/7 .$
- $2x + y - z = 2.$
 - $x + y - z = 3.$
 - $x - y = -1.$
- $6/7.$
- $t \rightarrow (t, t, t)$ or $t \rightarrow (-t, -t, -t).$
- Yes.
 - No.
 - Yes. The answer is yes if there is a constant, non-zero vector which is orthogonal to \mathbf{v} at each time t . Otherwise, the answer is no. For a), consider $(-1, 0, 5)$ and for c), consider $(0, 7, 1)$. No such vector exists for b) since in this case, $\mathbf{v}(0) = (0, 0, 1)$ and so such a vector would have to lie in the x - y plane. But then it couldn't be simultaneously orthogonal to $\mathbf{v}(\pi/2)$ and $\mathbf{v}(-\pi/2)$.
- In the first case, $L = 20x + y - z - 9$. In the second, $L = 20x - y + 3z - 7$.
 - In the first case, $L = z$. In the second, $L = 3y + z$.

13. a) In the first case, the plane is where $z = 1$. In the second, it is where $x + z = 1$.
 b) In the first case, the plane is where $2x - y = 2$. In the second, it is where $x + y - z = 1$.
14. $L = 2x - 5y + z$.
15. a) 2. b) 1. c) $-\sqrt{2}$.
16. $\nabla g = (-2, -4)$.
17. a) $\nabla f = (-\sin(x), 2y)$ so possible the critical points have the form $(n\pi, 0)$ with n an integer. The second derivative tests finds $(n\pi, 0)$ a local minimum when n is odd and a saddle when n is even.
 b) $\nabla f = (-\sin(x)\sin(y), \cos(x)\cos(y))$ so the critical points have the form $(n\pi, (m + 1/2)\pi)$ and $((n + 1/2)\pi, m\pi)$ where n and m are integers. In the first case, if both n and m are odd or both are even, it is a local maximum. If one is odd and the other not, then it is a local minimum. In the second case, it is a saddle for all n and m . (Use the 2nd derivative test.)
 c) $\nabla f = (2x, 3y^2 - 3)$ so the critical points are $(0, \pm 1)$. The point $(0, 1)$ is a local minimum and the point $(0, -1)$ is a local minimum.
18. The origin is the only interior critical point and it is a saddle. The extreme points on the boundary occur at $(\pm 3/\sqrt{2}, \pm\sqrt{2})$. The points $(3/\sqrt{2}, \sqrt{2})$ and $(-3/\sqrt{2}, -\sqrt{2})$ are maxima and the others are minima.
19. The maxima occur at $(\pm 1, 0, 0)$ and the minima at $(0, \pm 1, 0)$.
20. The maximum is at $(1, 0, 0)$.
21. The closest points are $(0, \pm 1, \pm 1)$ with distance $\sqrt{2}$.
22. The square of the length of \mathbf{E} is $x^4 + 4y^2 + z^2 + z^2y^2 - 2zy + 1$ which is smallest at the origin.
23. $3(\pi/2 - 1)$.
24. $4/(3\pi)$. (Remember to divide the integral of x by the area.)
25. $\pi/8$.
26. $\sin(1)/2$.

27. $\pi/2$

28. $\pi \sin(1)$.

29. $4\pi (1 - \cos(1))/3$.

30. $\int_0^1 \left(\int_0^r \left(\int_0^\pi (r^3 \sin(\theta) \cos^2(\theta)) d\theta \right) dz \right) r dr = 1/9$.

31. $\int_0^{\pi/2} \int_0^{\pi/4} \left(\int_0^{1/\cos(\phi)} (\rho^4 \sin(\theta) \cos(\theta) \sin^2(\phi) \cos^2(\phi)) \rho^2 d\rho \right) \sin(\phi) d\phi \int_0^\pi d\theta = 1/56$.

32. $\int_{-3/5}^{3/5} \int_{-\sqrt{9/25-x^2}}^{\sqrt{9/25-x^2}} \int_{\frac{4}{3}\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz dy dx =$
 $\int_0^{4/5} \left(\int_{\frac{3}{4}r}^{\sqrt{1-r^2}} \left(\int_0^{2\pi} d\theta \right) dz \right) r dr =$
 $\int_0^1 \left(\int_0^{\text{Arc tan}(4/3)} \left(\int_0^{2\pi} d\theta \right) \sin(\phi) d\phi \right) \rho^2 d\rho = 4\pi/15$.

33. a) $\int_0^\infty \int_0^{2\pi} e^{-r^2} d\theta r dr = \pi$.

b) The square of this integral is the Cartesian coordinate version of the preceding integral, so the integral in question equals $\pi^{1/2}$.

34. $u(t, x) = e^t (1 + (x - 4t)^2)^{-1}$.

35. a) Advection. b) Advection. c) Diffusion.

36. 0

37. $-2/\pi - 2/7$

38. 0.

39. a) $\mathbf{X}(u, v) = (2(1 - u^2/9 - v^2/25)^{1/2}, u, v)$ for values of (u, v) with $u^2/9 + v^2/25 \leq 1$.

b) $\mathbf{X}(u, v) = (u, 3(1 - u^2/4 - v^2/25)^{1/2}, v)$ for values of (u, v) with $u^2/4 + v^2/25 \leq 1$.

c) $\mathbf{X}(u, v) = (1 - u^2 - v^4, u, v)$ for values of (u, v) with $u^2 + v^4 \leq 1$.

d) $\mathbf{X}(u, v) = (u, v, 2 - u - v)$ for values of (u, v) in $u \geq 0, v \geq 0$ and $u + v \leq 2$.

40. a) $\int_{-5}^5 \int_{-3\sqrt{1-v^2/25}}^{3\sqrt{1-v^2/25}} \sqrt{1+4(u^2/81+v^2/625)/(1-u^2/9-v^2/25)} du dv.$

b) $\int_{-5}^5 \int_{-2\sqrt{1-v^2/25}}^{2\sqrt{1-v^2/25}} \sqrt{1+9(u^2/16+v^2/625)/(1-u^2/4-v^2/25)} du dv.$

c) $\int_{-1}^1 \left(\int_{-\sqrt{1-v^4}}^{\sqrt{1-v^4}} \sqrt{1+4u^2+16v^6} du \right) dv.$

41. $10/3$. (The area of the surface is π and the integral of z over the surface is $10\pi/3$.)

42. $\sin(1)$.

43. $500\pi/3$.

44. $(x, -y, 0)$.

45. a) $(x^2yz/2, 0, 0)$.

b) $(2z, 3x, y)$.

46. a) $(0, x/\pi)$.

b) $(x, 0, 0)$.

47. a) No such vector field exists because $(x, -2y, xy)$ has divergence -1 and the divergence of a curl is zero.

b) $(xy \cos(yz^2), 0, 0)$.

48. $5x + 3y = 0$.

49. $\iint_S x g dS$. Let $\mathbf{E} = (g, 0, 0)$; here $\text{div}(\mathbf{E}) = f$ while $\mathbf{E} \cdot \mathbf{n} = x g$ on the surface of the volume, V , in question. Thus, the divergence theorem implies that $\iint_S x g dS = \iiint_V f dV$.

50. a) 2π . This is a direct computation: Parameterize the circle by $t \rightarrow (\cos(t), \sin(t))$ and then path integral is just the integral of dt between 0 and 2π .

b) 2π . Use Green's theorem for a region with holes and note that the vector field in question has zero 'curl' in the sense that when written as (P, Q) , then $Q_x - P_y = 0$.

c) 0 . This is also a direct application of Green's theorem.

51. $\mathbf{A} = (-1, 0, 0, 0, 3)$, $\mathbf{B} = (-1, -1, 0, 2, 2)$.

52. The mean is $m = (n_1 m_1 + n_2 m_2)/(n_1 + n_2)$ and the standard deviation is

$$s = ((n_1 - 1) s_1^2 + (n_2 - 1) s_2^2)^{1/2} / (n_1 + n_2 - 1)^{1/2}.$$

53. For 3.18: This is $\Pr(B \cup C) = \Pr(B) + \Pr(C) - \Pr(B \cap C) = .099$
 For 3.19: This is $\Pr(A \cup B \cup C) = .0143$.
 For 3.20: This is $\Pr(A \cap \bar{B} \cap \bar{C}) + \Pr(\bar{A} \cap B \cap \bar{C}) + \Pr(\bar{A} \cap \bar{B} \cap C) = .1368$
 For 3.21: This is $\Pr(\text{affected person is female}) = .677$.
 For 3.22: This is $\Pr(\text{both affecteds are female}) = .263$.
 For 3.23: This is $\Pr(\text{both} < 80) = .160$.
54. For 3.85: Use Baye's theorem to find $\Pr(Y_1 | (X_1 \cap X_6 \cap X_4)) = .009$.
 For 3.87: You are computing $\Pr(X_7 | Y_2) = .7$.
 For 3.88: You are computing $\Pr(\bar{X}_7 | \bar{Y}_2) = .605$.
55. For 3.104: .938.
 For 3.105: .988.
56. For 4.37: Use the binomial distribution to find that the answer is .172.
57. For 4.44: The probability of the 82 year old dying in the next year is $p_1 = (l_{82} - l_{83})/l_{82} = .104$.
 Similar probabilities, $\{p_j\}_{2 \leq j \leq 11}$ can be obtained for the others. The sum, $\sum_{1 \leq j \leq 11} p_j = .176$, is the answer.
58. For 4.69: Use the binomial expansion with $n = 5, p = .4$ to find $\Pr(X = 3) = .230$.
 For 4.70: $\Pr(X \geq 3) = .317$ using binomial table (Table 1) in Chapter 4.
59. For 4.78: Use the Poisson distribution with $\mu = 15.6$ to find the answer $\cong 7.651 \times 10^{-13}$.
60. For 5.31: This is given by $\Phi(-1.667) = 1 - \Phi(1.667) \cong .048$.
 For 5.32: This is given by $\Phi(-3) = 1 - \Phi(3) \cong .0013$.
61. For 5.61: This is $84! / (29! \times 55!) (.24)^{29} (.76)^{55} \cong .009$
 For 5.62: Use the fact that $\Pr(X \geq 29) \cong \Pr(Y \geq 28.5)$ where Y is normally distributed with mean $= np = 20.16$ and variance $= npq = 15.32$. Thus, $\Pr(Y \geq 28.5) \cong .017$.
62. For 5.64: $\Pr(X \geq 90) = 1 - \Phi(2.307) = .0105$.
 For 5.65: Approximate the binomial distribution with a normal one of mean $np = 21.1$ and variance $npq = 20.8$. Then, $\Pr(X \geq 25) \cong 1 - \Phi(.0755) = .225$.
63. a1) $A' \cap C$ and it has size 3750.
 a2) $A \cap B' \cap C'$ and it has size 650.
 b) $A \cap D \subseteq C' \cap B$.

c) Probability = $\frac{600+3,750}{10,000} = 0.435$

64. a) $\binom{13}{5} = 13 \times 11 \times 9.$

b) $\binom{13}{5} \times \binom{13}{3} \times \binom{13}{3} \times \binom{13}{2} = 13 \times 11 \times 9 \times (13 \times 11 \times 2)^2 \times 13 \times 6.$

c) $12 \times$ (Answer to Part b).

65. a) To parse the logic, imagine that the Course Head labeled the easy exams E1 and E2. The possibilities for event B are:

H1 H2 E1 E2

H1 H2 E2 E1

H2 H1 E1 E2

H2 H1 E2 E1

E1 H1 E2 H2

E2 H1 E1 H2

E1 H1 H2 E2

E2 H1 H2 E1

Each of these cases has probability $\frac{1}{24}$ since there are 24 possible ways to distribute the exams. Without distinguishing the easy exams, then the cases are

H1 H2 E E

H2 H1 E E

E H1 E H2

E H1 H2 E

Each of these has probability $\frac{1}{12}$ since each corresponds to two distinct cases in the first list above.

b) The conditional probability of A occurring given B is that of obtaining either of the first two distributions in the second list above out of the four equally probable distributions. Thus, this probability is $\frac{1}{2}$.

c) This is the conditional probability of A occurring given B, so equals $\frac{1}{2}$.

d) Suppose that the student selects envelope 3. One is asked to compute here the conditional probability for envelope 3 to have a hard exam given that event B occurs. As indicated by the second list, there is one way for this to happen (the final distribution) out of four equally likely possibilities for event B. Thus, the probability is $\frac{1}{4}$.

66. Taking 4 courses, the probability of 2 A's is $6 \times (\frac{1}{2})^2 \times (\frac{1}{2})^2 + 4 \times (\frac{1}{2})^3 \times \frac{1}{2} + (\frac{1}{2})^4 = \frac{11}{16}$.
Taking 3 courses, the probability is $3 \times (\frac{2}{3})^2 \times \frac{1}{3} + (\frac{2}{3})^3 = \frac{20}{27}$, so 3 courses is better.

67. a) The probability that $Y > 1$ is the probability that $\pi(X - \frac{1}{2}) > \frac{\pi}{4}$ which is that of $X > \frac{3}{4}$.
This is $\frac{1}{4}$.

b) The probability that $Y < t$ is that of $X < \frac{1}{2} + \frac{1}{\pi} \arctan(t)$, so $F_Y(t) = \frac{1}{2} + \frac{1}{\pi} \arctan(t)$. The corresponding density function is the derivative of $F_Y(t)$, namely $\frac{1}{\pi(1+t^2)}$.

68. a) The region corresponds to the portion of the unit square in the X-Y plane that lies above the parabola $Y = X^2 - \frac{1}{4}$. The corresponding probability is the area of this region, $\frac{5}{6}$.

b) The expectation of Z is the integral $\int_0^1 \int_0^1 \frac{x^2}{4y+1} dy dx = \frac{1}{3} + \frac{1}{4} \ln(5)$.

69. The result is true by assumption for $n = 2$. Now, assume that the result is true for all integers less than some integer $n > 2$. Then, write $P(A_1 \cup \dots \cup A_n) = P((A_1 \cup \dots \cup A_{n-1}) \cup A_n)$ and invoke the $n = 2$ case using $A = A_1 \cup \dots \cup A_{n-1}$ and $B = A_n$. This then proves the equality $P(A_1 \cup \dots \cup A_n) = P(A_1 \cup \dots \cup A_{n-1}) + P(A_n)$. Now invoke the induction hypothesis to write $P(A_1 \cup \dots \cup A_{n-1})$ as the sum of the probabilities of the A_k 's for $k \leq n-1$ to complete the proof for the integer n case.

70. a) False. (Take the case $A = B$.)

b) False. (The number is 13.)

c) False. (The probability is $\frac{11}{36}$.)

d) True. (As the expected number is $7/6$, the only issue is whether 0 sixes has larger probability than 1. This isn't the case as the former has probability $(\frac{5}{6})^7$ and the latter $\frac{7}{6}(\frac{5}{6})^6$.)

e) True. (The collection of all finite subsets is countable.)

f) False. (For each positive integer n , let A_n denote the set of weights w where $2^{-n} < p(w) \leq 2^{-(n-1)}$. Then, the number of weights in A_n is at most 2^n . As the set of weights with $p(w) > 0$ is $\cup_n A_n$, a countable union of countable sets, this set is countable.)

g) False. (The probability is $F(b, d) - F(a, c) - F(b, d) + F(a, c)$.)

71. a) The events are not independent since $P(A \cap B) = 0.3$ while $P(A) \times P(B) = \frac{6}{25}$.

b1) $A \cap B^c$. The probability is $P(A) - P(C) = 0.1$.

b2) $A \cup B$. The probability is $P(A) + P(B) - P(C) = 0.7$.

c) $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{3}{4}$.

d) $P(A^c|B^c) = \frac{P(A^c \cap B^c)}{P(B^c)} = \frac{1 - P(A \cup B)}{1 - P(B)} = \frac{3}{4}$.

72. a) $4 \times \binom{13}{4} = 13 \times 11 \times 10 \times 2$.

b) 13×4 .

c) $13 \times 12 \times 6 \times 3$.

73. $\binom{4}{2} \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2 = \frac{25}{216}$.

74. The sum of the probabilities is greater than 1.

75. a) $e^{-2} - e^{-3}$.

b) The probability that Y is less than t is that of $X > e^{-t}$, so Y has distribution function $1 - e^{-t}$.
Thus, the density function for Y is e^{-t} .

c) The expectation of Y is the integral of $t e^{-t}$ from 0 to ∞ . This is 1.

76. a) $\frac{1}{20}$

b) $\frac{1}{10}$

c) The possible cake numbers for S1 and S2 are (1, 5), (5, 1), (5, 2), (5, 3), (5, 4).

d) $\frac{2}{5}$.

e) $\frac{1}{5}$.

f) $\frac{4}{5}$.

77. a) 13%.

b) $\frac{4}{13}$

c) $\frac{2}{29}$