Math 21a second practicy exam for Hourly 2

What follows is a model of the second hourly exam (it is more or less the exam given last fall in this course). The upcoming exam on Wednesday, November 15 will be roughly similar to this one in length and difficulty. To study for the upcoming exam, first work the suggested problems for the Chapters 2, 3.1-3.2 and 4.4 and in the Supplement on Lagrange Multipliers. The exam will cover solely material from these readings. The list of suggested problems is posted on the course website. You should also work the practice problems for Hourly 2 supplied elsewhere at this website. After you have worked the suggested problems and checked your answers with those in the book, and worked the practice problems and checked your answers, try taking this exam as practice for the real thing. In this regard, note that you will have two hours for the real thing, but we hope that most people finish under the two hour limit. The answers to this practice hourly are provided elswhere at the website.

By the way, in the real exam, each problem will be printed on a separate page and you will be asked to provide your answers on that page and to use the back of that page if you need more space to show your work.

Remember that the exam on *Wednesday, November 14* is from *7:30-9:30pm* in *Lecture Halls C and E* of the *Science Center*. Please come a few minutes early as the exam will start promptly at 7:30pm.

1) __ 2) __ 3) __ 4) __ 5) __ 6) __ : Total ______

Name: ____

Circle the name of your Section TA:

Arinkin(10)•Arinkin(12)•Bamberg•Cornut•Kaplan•Karu•Knill•Libine•Liu•Taubes•Williams

Instructions:

- Print your name in the line above and circle the name of your section TF.
- Answer each of the questions below in the space provided. If more space is needed, use the back of the facing page.
- Please give justification for answers if you are not told otherwise.
- Please write neatly. Answers which are deemed illegible by the grader will not receive credit.
- No calculators, computers or other electronic aids are allowed; nor are you allowed to refer to any written notes or source material; nor are you allowed to communicate with other students. Use only your brain and a pencil.
- You have 2 hours to complete your work.
- Note that vectors are indicated below by bold face type.

In agreeing to take this exam, you are implicitly agreeing to act with fairness and honesty.

1. Suppose that f(x, y) is a function with the following properties:

   - f(1, 2) = 6.
   - The directional derivative at (1, 2) of f in the direction of **u** = $(\frac{3}{5}, \frac{4}{5})$ is 5
   - The directional derivative at (1, 2) of f in the direction of **v** = $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is $\frac{1}{\sqrt{2}}$.

   Given this information, answer the following questions:

   a) Write the gradient of f at (1, 2).
   b) Write an equation for the tangent plane to the graph of f at the point (1, 2, 6).
   c) Estimate the value of f(1.02, 2.05).

2. Let a, b and c be non-negative constants and F the function

$$F(x, y, z) = - x \ln(x) - y \ln(y) - z \ln(z) - a x - b y - c z.$$

Find the stationary points of F where $x > 0$, $y > 0$, $z > 0$ and $x + y + z = 1$. Don't search for stationary points where any of x, y or z are zero.

(If you are interested in where this question comes from, here is the answer: This sort of extremal problem arises when predicting the shapes of biological molecules. For example, proteins are complicated molecules that often fold into a number of stable shapes. Moreover, the biological effect of the protein is very sensitive to the particulars of its shape. If a certain protein has three possible folded shapes which we label as X, Y and Z, then in a given environment, a fraction, x of the protein molecules will have shape X, a fraction y will have shape Y and a fraction z will have shape Z. It is a fundamental problem in biochemistry to predict these fractions x, y and z. In particular, these fractions are known to depend on the energies, a, b and c of the shapes X, Y and Z; and the correct values for x, y and z are the constrained minima of $F(x, y, z) = -x \ln(x) - y \ln(y) - z \ln(z) - a x - b y - c z$ where the constraint is $x + y + z = 1$. In this context, the function F is called the *free energy*.)

3. Integrate the function $f(x, y) = x$ over the interior of the region where $x \geq 0$ and $x^2/4 + y^2 \leq 1$.

4. Let $f(x, y) = x^4 - 2 x^2 + 3 y^3 - 9 y$.
   a) Find all stationary points of f.
   b) Classify each of the points found in Part a as local maxima, minima or saddle.
   c) Find all points on the level set $f = 2$ where the tangent line is parallel to the y axis.

5. Evaluate $\int_0^1 \left(\int_0^{\operatorname{Arcsin}(x)} dy \right) dx$.

6. A rectangular, open top aquarium with a slate bottom and glass side is to hold 20,000,000 cubic centimeters of water. Suppose that glass cost 10 cents per square centimeter and slate cost 50 cents per square centimeter. Find the dimensions (height, length and width) which minimize the price of the aquarium. Please justify your answer.