

Math 21a Handout on Lagrange Multipliers - Spring 2000

The principal purpose of this handout is to supply some additional examples of the Lagrange multiplier method for solving constrained equations for three unknowns. In this regard, remember that the basic problem is to find the maxima, minima and/or stationary points of some given function $f(x, y, z)$ on $\mathbf{R}^3$ subject to the constraint that $g(x, y, z) = 0$ where g is another function.

Note that these constrained problems typically arise in one of two ways: First, they can arise when looking for the extreme points of a function in some region of space; for in this case a search must be made for extreme points both inside the region and on its boundary. In particular, the problem of finding the extreme points on the boundary can be phrased as a constrained extremal problem. For example, imagine that the region in question is the solid Earth, and the function in question gives the temperature at each point. Then, a search for extreme points must look for interior extreme points and also extreme points on the surface.

Constrained extremal problems also arise when looking for the stationary points of a given function's restriction to some given surface in $\mathbf{R}^3$. For example, take the surface in question to be the surface of the earth, and consider finding the extreme points of the function whose values measure a year's average of sun light intensity at each point.

Anyway, what follows are some sample constrained problems.

Example 1: Model the surface of the Earth as the sphere where $x^2 + y^2 + z^2 = 1$. Then, suppose that the average temperature at each point on the surface is given by the values of the function $T = x^2 + 2xy + y^2 - 2z^2$. Find the points with the maximum and also the minimum temperature.

Here is the solution: The constraint function is $g(x, y, z) \equiv x^2 + y^2 + z^2 - 1$, and the extreme points occur where $\nabla T = \lambda \nabla g$ for some constant λ . A computation finds:

$$\begin{aligned}\nabla g &= (2x, 2y, 2z) \\ \nabla T &= (2x + 2y, 2y + 2x, -4z)\end{aligned}\tag{1}$$

Thus, the equation $\nabla T = \lambda \nabla g$ holds if and only if

$$\begin{aligned}2x + 2y &= \lambda 2x \\ 2y + 2x &= \lambda 2y \\ -4z &= \lambda 2z\end{aligned}\tag{2}$$

The last point in (2) can hold only if $\lambda = -2$ or $z = 0$. I'll treat these two possibilities in turn.

Consider the case $\lambda = -2$. Then, the first and second points in (2) read

$$\begin{aligned}2x + 2y &= -4x \\ 2y + 2x &= -4y\end{aligned}\tag{3}$$

The latter points can be rephrased to read

$$\begin{aligned}6x &= -2y \\ 6y &= -2x\end{aligned}\tag{4}$$

This first line says that $x = -y/3$, while the second says that $x = -3y$. Clearly, these two assertions are compatible only if $x = y = 0$. Finally, this last forces $z = \pm 1$ since $x^2 + y^2 + z^2 = 1$. Thus, I have found two stationary points, $(0, 0, 1)$ and $(0, 0, -1)$.

Now consider the case where $z = 0$. In this case, the first two points in (2) can be rephrased to read

$$\begin{aligned}(1 - \lambda)x &= -y \\ (1 - \lambda)y &= -x\end{aligned}\tag{5}$$

Now, it is not possible for λ to equal 1 since then (5) says that $x = y = 0$ and with $z = 0$, this would violate the constraint condition $x^2 + y^2 + z^2 = 1$. Then, as $\lambda \neq 1$, the first point in (5) implies that $x = -y/(1 - \lambda)$. Meanwhile, the second point gives $x = -(1 - \lambda)y$. Now, you might think that these two conditions on x can be solved by setting $y = 0$; but then x would have to vanish too and with $z = 0$, the constraint $x^2 + y^2 + z^2 = 1$ would be violated. So with $y \neq 0$, the only way to solve these constraints is if $1/(1 - \lambda) = (1 - \lambda)$. This last equation holds if and only if $\lambda = 0$ or $\lambda = 2$. In the case $\lambda = 0$, both points of (5) read $x = -y$. Since $z = 0$, this last condition plus the constraint $x^2 + y^2 = 1$ force $(x, y) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ or $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. On the other hand, in the case $\lambda = 2$, both points of (5) read $x = y$ and, since $z = 0$, the constraint $x^2 + y^2 = 1$ implies that $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

To summarize: There are six stationary points for this extremal problem:

$$(0, 0, 1), (0, 0, -1), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)\tag{6}$$

Substituting these values into the function T shows that the first two are minima and the last two are maxima. The middle pair are stationary points that are neither local minima nor local maxima.

Example 2: Consider the problem of finding local maxima and minima of $f(x, y, z) = x^2 - y^2 + z^2/(1 + z^2)$ on the infinite cylinder where $x^2 + y^2 = 1$ and $-\infty < z < \infty$. In this example, the constraint function is $g(x, y, z) = x^2 + y^2 - 1$ and the components of the Lagrange multiplier equation $\nabla f = \lambda \nabla g$ read

$$\begin{aligned}2x &= \lambda 2x \\ -2y &= \lambda 2y \\ 2z/(1 + z^2)^2 &= 0\end{aligned}\tag{7}$$

Here is how to analyze (7): First off, the last point in (7) requires $z = 0$. Next, observe that the first point in (7) requires either $x = 0$ or $\lambda = 1$. If $x = 0$, then the constraint $x^2 + y^2 = 1$ forces $y = \pm 1$, and thus $(0, 1, 0)$ and $(0, -1, 0)$ are stationary points. On the other hand, if $x \neq 0$, then $\lambda = 1$ and the second point requires $y = 0$. In this case, the constraint $x^2 + y^2 = 1$ forces $x = \pm 1$. Thus, $(1, 0, 0)$ and $(-1, 0, 0)$ are also stationary points. Hence, there are four stationary points in all. Substituting these into the expression for f finds that $(0, \pm 1, 0)$ are local minima while $(\pm 1, 0, 0)$ are saddles.

Example 3: In this case, the consider the problem of finding all stationary points of the function $f(x, y, z) = -x^2 + z^2$ on the paraboloid given by the constraint $g(x, y, z) = z - x^2 - y^2 = 0$. In this case, the three components of the Lagrange multiplier equation $\nabla f = \lambda \nabla g$ read

$$\begin{aligned}-2x &= -\lambda 2x \\ 0 &= -\lambda 2y \\ 2z &= \lambda\end{aligned}\tag{8}$$

To find the solutions to (8), note that the middle point requires either $\lambda = 0$ or $y = 0$. In the former case, the third point requires $z = 0$ and the first requires $x = 0$. Then, the constraint equation $z = x^2 + y^2$ forces $y = 0$. So, $(0, 0, 0)$ is a stationary point.

Now consider the case $y = 0$ solving the middle point in (8). The first point in (8) requires either $x = 0$ or $\lambda = 1$. The former possibility gives the point $(0, 0, 0)$ as a stationary point (again) since $x = y = 0$ and the constraint equation $z = x^2 + y^2$ forces z to vanish too. On the other hand, if $x \neq 0$ and $\lambda = 1$, then the

third point in (8) requires $z = 1/2$. Since $y = 0$, the constraint equation reads $z = x^2$ so $x = \pm \frac{1}{\sqrt{2}}$. Thus, $(\frac{1}{\sqrt{2}}, 0, \frac{1}{2})$ and $(-\frac{1}{\sqrt{2}}, 0, \frac{1}{2})$ are also stationary points.

Plugging into $f = -x^2 + z^2$ the values of x and z from these three stationary points finds that $(\pm \frac{1}{\sqrt{2}}, 0, \frac{1}{2})$ are global minima, while $(0, 0, 0)$ is neither a local minimum or maximum. (There is no global maximum since f can be made arbitrarily large by taking x large and $z = x^2$.)

Homework Problems:

- 1) Find the minimum distance from the surface $x^2 + y^2 - z^2 = 1$ to the origin.
- 2) Find the maximum and minimum values of $x + y - 2z$ on the sphere $x^2 + y^2 + z^2 = 1$.
- 3) Find the maximum and minimum values of the function $f(x, y, z) = xyz$ on that part of the surface $x + y + 4z = 36$ in the first octant, i.e. where $x, y, z \geq 0$.
- 4) Model the earth as the sphere $x^2 + y^2 + z^2 = 1$. Suppose that the temperature at a point (x, y, z) on the surface is $T(x, y, z) = x^2 - y^2 + z + 1$ in appropriate units. Find the points with the highest and lowest temperatures.
- 5) Find the maximum and minimum values of the function $f(x, y) = 2x + y$ on the ellipse $2x^2 + 2xy + 4y^2 = 98$.
- 6) Suppose that production at a lollipop factory is modelled by the Cobb-Douglas function

$$p(x, y) = 500x^{0.4}y^{0.8}$$

where x is the number of units of labor and y is the number of units of capital. If the labor costs \$60 per unit and the cost of capital is \$100 per unit, find the least costly combination of inputs to produce 10,000 units of lollipops and find the minimum cost.

- 7) Design an open top cylindrical container which holds 50cc and has the minimum possible surface area. What should its dimensions be?
- 8) Use any method you like to find the point on the ellipsoid with equation $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{4} = 1$ that is closest to the plane with equation $2x + 3y + z = 25$.
- 9) Find the dimensions of the rectangular box of largest volume, with sides parallel to the coordinate axes, that can be inscribed in the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{4} = 1$.
- 10) You have 36 square units of material with which to make a box. You can construct either a rectangular box or a cylindrical box. Both have a bottom but no top. If you make the box in either case which contains the most volume, which one will give the greater volume and by how much?

- 11) A cookie manufacturer finds that production depends on three factors: the number of units x of labor, the number of units y of capital, and the number of elves z working behind the scenes and conducting magic. The production P depends on these factors according to the relation

$$P = 300x^{0.4}y^{0.5}z^{0.1}$$

It costs the company \$80 per unit of labor, \$25 per unit of capital, and \$15 per elf. The company is required to meet a production P of 12,000 boxes of cookies. What choices of x , y , and z will meet this production at the lowest possible cost?

- 12) Suppose that production (measured in lollipops) at a small lollipop factory is modeled by the Cobb-Douglas function

$$p(x, y) = 500x^{0.7}y^{0.5}$$

where x is the number of units of labor and y is the number of units of capital.

Further suppose that the cost of labor is \$35 per unit and the cost of capital is \$16 per unit.

- Find the least costly combination of labor and capital needed to produce 40,000 lollipops and find this minimum cost.
 - Now find what combination of labor and capital will yield the maximum number of lollipops with a fixed budget of \$4800 and find this maximum number of lollipops.
- 13) Production at a certain factory is determined by the function $P(x, y) = 300x^{\frac{1}{2}}y^{\frac{1}{3}}$ where x is the number of units of labor and y is the number of units of capital. We know that the unit prices for labor and capital are \$100 and \$120 respectively. The managers at the factory have been raising their budget to pay for more labor and capital according to the relations: $\begin{cases} x(t) = 20 + 8t \\ y(t) = 45 + 8t \end{cases}$ where t is measured in months. After 10 months the owners start to raise questions about the efficient operation of the factory.
- Determine how much money is being spent at $t = 10$ months and what the production is at that point.
 - At that point, at what rate would the production increase if we were to spend an additional dollar on labor?
 - Again at that point, at what rate would production increase if we were to spend an additional dollar on capital?
 - For the same budget that you computed in part a), how would you reallocate your resources, i.e. what should x and y be, if your goal is to get maximal production for the same budget?
 - Now that you've reallocated resources for greatest efficiency, if you were to spend an additional dollar in the most efficient way, at what rate would the production increase?