Math 21a Physics Supplement - Part I

1) Work and Energy

To simplify the notation, we introduce the following shorthand:

$$\frac{d}{dt}\mathbf{r} \equiv \dot{\mathbf{r}}\,, \qquad \frac{d^2}{dt^2}\mathbf{r} \equiv \ddot{\mathbf{r}} \tag{1}$$

Newton's law asserts that the position vector **r**(t) of a particle of mass m under the influence of a force **F** obeys the equation

$$\mathbf{F} = m\ddot{\mathbf{r}} \tag{2}$$

a) Work

Suppose that the components of the force vector **F** do not depend on time or the position of the particle. Thus, **F** has components (a, b, c) which are numbers, not functions.
(For example, **F** = (1, 2, 3).) Then, the work done in moving the particle from position $\mathbf{r}_0$ to position $\mathbf{r}_1$ is (by definition)

$$W \equiv \mathbf{F} \bullet (\mathbf{r}_1 - \mathbf{r}_0). \tag{3}$$

Note that this notion of work can be generalized to apply to any force vector **F**, constant or not; but we are not ready at this point in the course for the generalization.

b) Energy

When subject to a constant force, the energy of a particle at position **r** with velocity vector $\dot{\mathbf{r}}$ is the function

$$E = \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 - \mathbf{F}\cdot\mathbf{r} \tag{4}$$

Note that Newton's law (Equation (2)) implies that the energy function does not change as time evolves. Indeed, we can differentiate (4) to find that

$$E = m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} - \mathbf{F}\cdot\dot{\mathbf{r}} = \mathbf{F}\cdot\dot{\mathbf{r}} - \mathbf{F}\cdot\dot{\mathbf{r}} = 0 \tag{5}$$

where the second equality comes by substituting for $m\ddot{\mathbf{r}}$ using Equation (1).

Note that the constant force vector is not the only kind of force for which energy can be defined and for which the energy is independent of time along the trajectory. Consider, for example the following case: Let $f(r)$ be a function of the distance, $r \equiv |\mathbf{r}|$, of the particle from the origin, and consider the force $\mathbf{F} = f(r)\,\mathbf{r}/r$. For example, if the force is gravity on a mass m due to a mass M at the origin, then $f(r) = -\frac{GMm}{r^2}$, where G is the Gravitational constant. In any event, when $\mathbf{F} = f(r)\,\mathbf{r}/r$, Newton's law reads

$$m\ddot{\mathbf{r}} = f(r)\frac{\mathbf{r}}{r} \tag{6}$$

In this case, the energy is defined to be

$$E = \tfrac{1}{2} m|\dot{\mathbf{r}}|^2 + V(r) \qquad (7)$$

where -*V*(*r*) is an anti-derivative of *f*(*r*). That is, $-\frac{dV}{dr} = f$. For the case where $f(r) = -\frac{GMm}{r^2}$, this function *V*(*r*) (called the <u>potential</u>) can be taken to be $V(r) = -\frac{GMm}{r}$.

In any case, with **F** = *f*(*r*) **r**/r, the energy is also constant along a particle's trajectory as can be seen by first differentiating and using the Chain rule to find that

$$\dot{E} = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} - f(r)\dot{r}\ . \qquad (8)$$

One then employs the formula $\dot{r} = \dot{\mathbf{r}} \cdot \frac{\mathbf{r}}{r}$ (which is also an application of the Chain rule together with the fact that $r = (\mathbf{r} \cdot \mathbf{r})^{1/2}$). This allows (8) to be written as

$$\dot{E} = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} - f(r)\dot{\mathbf{r}} \cdot \frac{\mathbf{r}}{r}\ . \qquad (9)$$

Finally, use Equation (6) to replace $m\ddot{\mathbf{r}}$ in this last equation by *f*(*r*) **r**/*r* to see that $\dot{E}$ vanishes.

2) Planetary Motion

Suppose that an object (planet, asteroid, whatever) travels through space under the gravitational influence of a star. Newton's laws allow one to describe trajectory of the planet. This supplement describes the situation.

a) Newton's Laws

There is a natural coordinate system to describe the motion. This is the so called center of mass coordinate system. (See the final section, f.) In this coordinate system, the position vector **r**(t) for the object at time t (the vector from the origin to the planet's position) evolves in time according to a differential equation of the following form:

$$\ddot{\mathbf{r}} = -\frac{k\,\mathbf{r}}{r^3} \qquad (10)$$

Here, κ is a positive constant which is determined by the masses of the object and the star. As usual, $r = |\mathbf{r}|$ is the distance of the object from the origin.

b) The energy

The standard strategy for solving (10) is to find so called "constants of the motion". These are quantities which are constant along each trajectory. The energy,

$$E = \tfrac{1}{2} m|\dot{\mathbf{r}}|^2 - \frac{k\,m}{r} \tag{11}$$

is one such constant. To show that E is constant, consider its time derivative under the assumption that $\mathbf{r}(t)$ obeys (10). One finds that

$$\dot{E} = m(\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \frac{k\,\dot{\mathbf{r}} \cdot \mathbf{r}}{r^3}) \tag{12}$$

The fact that the latter is zero follows by taking the dot product of both sides of (10) with the vector $\dot{\mathbf{r}}$.

The derivation of (12) uses the fact that $r = (\mathbf{r}\cdot\mathbf{r})^{1/2}$ and that the time derivative of a dot product obeys

$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \dot{\mathbf{A}}\cdot\mathbf{B} + \mathbf{A}\cdot\dot{\mathbf{B}} \tag{13}$$

when **A** and **B** are both vector valued functions of time. (Equation (13) can be verified by writing out $\mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3$ and then differentiating.)

You should think about what it means if the energy E is constant along a trajectory. For example, suppose E is negative. Can the trajectory get arbitrarily far from the origin? No, in fact, according to (11),

$$\frac{mk}{r} = \frac{m|\dot{\mathbf{r}}|^2}{2} - E \geq -E\,. \tag{14}$$

If $(-E) > 0$, then multiplying both sides above by $r/(-E)$ doesn't change the direction of the inequality and produces

$$\frac{mk}{-E} \geq r. \tag{15}$$

Thus, if E is negative, the orbit is necessarily bounded.

What about when E is zero or positive? As we shall see, this case allows orbits which are unbounded.

c) The angular momentum

Another constant of the motion is the angular momentum,

$$\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}\,. \tag{16}$$

To see that this is independent of t when $\mathbf{r}(t)$ obeys (10), simply differentiate to find that

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times m\dot{\mathbf{r}} + \mathbf{r} \times m\ddot{\mathbf{r}} = \mathbf{0} + \mathbf{r} \times (\frac{-mk\,\mathbf{r}}{r^3}) = \mathbf{0} + \mathbf{0} = \mathbf{0}\,. \tag{17}$$

The derivation here uses the fact that the time derivative of the cross product of two time dependent vectors obeys

$$\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \dot{\mathbf{A}}\times\mathbf{B}+\mathbf{A}\times\dot{\mathbf{B}} \tag{18}$$

as can be verified by writing out $\mathbf{A}\times\mathbf{B} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$ and then differentiating. Also, the derivation of (17) uses the fact that $\mathbf{V}\times\mathbf{V} = \mathbf{0}$ for any vector $\mathbf{V}$.

By the way, the fact that $\mathbf{L}$ is constant means that each $\mathbf{r}(t)$ is at 90° to $\mathbf{L}$. Thus, the whole trajectory lies in the plane through the origin whose normal vector is $\mathbf{L}$. Of course, this assumes that $\mathbf{L} \neq \mathbf{0}$. If $\mathbf{L} = \mathbf{0}$, then this means that $\dot{\mathbf{r}}$ is proportional to $\mathbf{r}$ at each point. That is, $\dot{\mathbf{r}} = \alpha(t)\cdot\mathbf{r}$, where $\alpha(t)$ is some function of t. This means that the whole trajectory lies upon a line. Indeed, this last equation implies that the unit vector $\mathbf{r}/r$ is constant (when $r \neq 0$), as one can see by differentiating.

d) Reducing to the plane

To find the trajectory when $\mathbf{L} \neq 0$, it pays at this point to focus attention to the plane on which $\mathbf{r}$ lies. Now one plane is pretty much the same as any other, and with this understood, one can set up a coordinate system on the plane through the origin perpendicular to $\mathbf{L}$ plane with coordinates, say (u, v). Furthermore, it is convenient to introduce *polar coordinates* in this plane, by writing $u = r\cos q$ and $v = r\sin q$. With this understood, then r and q are functions of t along the trajectory. That is,

$$\mathbf{r} = (r\cos q,\ r\sin q) \quad\text{and}\quad \dot{\mathbf{r}} = (\dot{r}\cos q - r\dot{q}\sin q,\ \dot{r}\sin q + r\dot{q}\cos q)\ . \tag{19}$$

With respect to these coordinates, we find that $|\mathbf{L}| = |\lambda|$, where λ is a constant given by

$$\lambda = mr^2\dot{q}\ . \tag{20}$$

Meanwhile,

$$E = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{q}^2) - \frac{k\,m}{r}\ . \tag{21}$$

Now, Equations (20) and (21) can be used to solve for $\dot{r}$ and $\dot{q}$ in terms of r and the constants E and λ. Indeed, (20) directly gives $\dot{q} = \frac{l}{mr^2}$ and (21) with this last identity for $\dot{q}$ implies that

$$\dot{r} = \pm\sqrt{\tfrac{2E}{m} - \tfrac{l^2}{m^2r^2} + \tfrac{2k}{r}} \tag{22}$$

The preceding equation is a differential equation for $r(t)$ which can be integrated to give

$$\int \frac{dr}{\sqrt{\tfrac{2E}{m} - \tfrac{l^2}{m^2r^2} + \tfrac{2k}{r}}} = \pm t \tag{23}$$

e) The trajectory

The integral in (23) can be evaluated. However, it is also instructive to consider the trajectory as tracing out a path in the (*u*, *v*) plane. This path is described by giving the distance *r* as a function of the angle *q*. In this case, (22) and the equality $\dot{q} = \frac{l}{mr^2}$ imply the relation

$$\frac{dr}{dq} = \pm\sqrt{\tfrac{2E}{m} - \tfrac{l^2}{m^2r^2} + \tfrac{2k}{r}}\,\frac{mr^2}{l} \tag{24}$$

which can be shown to be the equation of a conic section with the origin as one of the focal points. In particular, the $E < 0$ case is an ellipse, the $E = 0$ case is a parabola and the $E > 0$ case is one branch of a hyperbola. (The eccentricity here is $(1 + 2E\lambda^2/\kappa^2)^{1/2}$.)

By the way, to see how a conic section arises from (24), make the substitution $w = \frac{l}{mr} - \frac{km}{l}$. Then, by virtue of (24), *w* obeys

$$-\frac{dw}{dq} = \pm\sqrt{\tfrac{2E}{m} + \tfrac{k^2m^2}{l^2} - w^2} \tag{25}$$

which can be solved (using integration formulas from Math 1b) by the further substitution $w = (\frac{2E}{m} + \frac{k^2m^2}{l^2})^{1/2}\cos(q - q_0)$ where q_0 is a constant of integration. Putting this back in terms of *r* gives the equation for the conic section,

$$r = \frac{\frac{l^2}{km^2}}{[1 + (1 + \frac{2El^2}{m^3k^2})^{1/2}\cos(q - q_0)]}\,. \tag{26}$$

f) The center of mass

As remarked at the outset, the vector **r** in (10) is the position vector for the object where the origin is at the center of mass of the system. Note that this center of mass point is not at the center of the sun. Indeed, just as the center of the planet moves due to the gravitational force from the sun, so the center of the sun moves with time due to the gravitational force from the planet. Thus, the center of the sun is not a point which is fixed in time. Anyway, in an ideal world run by Newton's laws, here is how to find the center of mass point, **c**: First, choose any <u>fixed</u> coordinate system, and let **x** denote the position vector of the planet and **X** that of the sun. Meanwhile, let *m* denote the mass of the planet and *M* that of the sun. Then, in this chosen coordinate system,

$$\mathbf{c} = \frac{m\mathbf{x} + M\mathbf{X}}{m + M}\,. \tag{27}$$

The center of mass coordinate system is then chosen so that $\mathbf{c} = \mathbf{0}$. Newton's laws insure that **c** remains zero for all time as long as the system is not acted upon by additional forces.

Note that in the center of mass coordinate system, $|\mathbf{X}| = \frac{m|\mathbf{x}|}{M}$, so that when the sun is much more massive than the planet, the center of mass is very close to (but not exactly) the center of the sun. In particular, this distance is relatively small when the planet in question has small mass; such is the case when the planet is Earth. However, the difference between these two points is not so small when the planet is large like Jupiter or Saturn. In particular, if you graduate from Harvard and go on to work for NASA designing probes to the outer planets, please use the correct origin for your coordinates, or else your multi-million dollar probe will miss its target.

By the way, the constant κ in (10) is given in terms of m, M and the gravitational constant G by the equation $\kappa = GM/(1 + m/M)^2$.

3) Torque and Angular Momentum

According to Newton, the position vector, $\mathbf{r}(t)$, of a particle changes with time t under the influence of a force vector $\mathbf{F}$ according to the rule

$$\mathbf{F} = m\ddot{\mathbf{r}} . \tag{28}$$

Here, m is the mass of the particle.

The <u>momentum</u> of the particle is, by definition, the vector $\mathbf{p} \equiv m\dot{\mathbf{r}}$. Thus, momentum is mass times velocity. The <u>angular momentum</u> is defined to be the vector

$$\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}} = \mathbf{r} \times \mathbf{p} . \tag{29}$$

Note that $\mathbf{L}$ is perpendicular to both the position vector $\mathbf{r}$ and the velocity vector $\dot{\mathbf{r}}$.

In some sense, angular momentum measures the deviation from motion on a straight line. Indeed if $\mathbf{L} = \mathbf{0}$, then the velocity vector $\dot{\mathbf{r}}$ is proportional to the position vector $\mathbf{r}$, and this implies that the particle travels along a straight line (but maybe back and forth). To see that the condition $\dot{\mathbf{r}}$ being proportional to $\mathbf{r}$ means straight line motion, differentiate the unit vector $\mathbf{r}/|r|$ under this assumption to see that the latter is constant.

In general, $\mathbf{L}$ will evolve in time if $\mathbf{r}$ does. Indeed, differentiate (29) to find that

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times m\dot{\mathbf{r}} + \mathbf{r} \times m\ddot{\mathbf{r}} = \mathbf{r} \times \mathbf{F} . \tag{30}$$

Here, the final equality comes by substituting (28) for $m\ddot{\mathbf{r}}$ while the second equality arises because $\dot{\mathbf{r}} \times \dot{\mathbf{r}} = \mathbf{0}$. The vector $\mathbf{r} \times \mathbf{F}$ is called the <u>torque</u>.

By the way, if $\mathbf{F}$ has the form $\mathbf{F} = f(r)\, \mathbf{r}/r$, then the angular momentum vector is constant since $\mathbf{r} \times \mathbf{F} = \mathbf{0}$. An example is the case where the force is due to the gravitational pull of a mass, M, at the origin, for here, $f(r) = -\, GmM/r^2$, where G is the gravitational constant.

Finally, note that if $\mathbf{L}$ is constant, then the motion takes place solely in a plane through the origin. Indeed, this follows because

(a) $\mathbf{r}$ is always perpendicular to $\mathbf{L}$ as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and
(b) the vectors perpendicular to a given vector define a plane.

(31)

For example, the trajectory of a planet orbiting the sun lies in a plane. (This last statement is one of Kepler's laws. Kepler based this 'law' on empirical data from observations of the sky, and Newton got justly famous for using his laws of motion to explain why Kepler's laws hold.)

4) Work and Energy - further observations

As mentioned at the end of Section 5.1 in the text, the <u>work</u> done in moving a particle along a path γ in $\mathbf{R}^3$ when a force vector $\mathbf{F}$ acts is defined to be

$$W(g) = \int_a^b \mathbf{F} \cdot d\mathbf{x} \tag{32}$$

This is to say that term 'work done in moving along γ' in physics has a specific meaning, the latter being the value of (32). (Presumably, this mathematical definition of work meshes well with our intuitive idea of work done.)

Let me remind you (from the definition in the text) that the shorthand on the right side of (32) has the following meaning: Parameterize the path γ by choosing an interval, [a, b] on the line, and a vector valued function $\mathbf{x}(t)$ for $a \le t \le b$ which traces out the path γ. With this done, then the right side of (32) is computed by the ordinary integral

$$W(g) = \int_a^b \mathbf{F}(\mathbf{x}(t) \cdot \dot{\mathbf{x}}(t)\, dt\ . \tag{33}$$

Now, suppose that the path γ is the path that the particle would have traveled were its trajectory γ given by Newton's law; thus $\mathbf{x}(t)$ obeys

$$\mathbf{F} = m\ddot{\mathbf{x}}\ . \tag{34}$$

In this case, substitution in (33) gives

$$W(g) = m\int_a^b \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}\, dt \tag{35}$$

which is nothing more than

$$W(g) = \tfrac{1}{2} m \int_a^b \frac{d}{dt} |\dot{\mathbf{x}}|^2 dt = (\tfrac{1}{2} m|\dot{\mathbf{x}}|^2)\,|_{t=b} - (\tfrac{1}{2} m|\dot{\mathbf{x}}|^2)\,|_{t=a}\ . \tag{36}$$

(The final equality in (36) is just the Fundamental Theorem of Calculus.) In any event, (36) demonstrates that the work done by a particle moving according to Newton's law is given by the difference between the squared norms of the velocities times the mass divided by 2.

This last formula is not unrelated to the notion of <u>conservation of energy</u>. In this regard, suppose that $\mathbf{F} = -\nabla V$, that is, the force is the gradient of a potential. (For example, the gravitational force from a point particle of mass M at the origin is the gradient of $V = -GmMr^{-1}$, where $r = |\mathbf{x}|$ is the distance to the origin.) Anyway, if $\mathbf{F} = \nabla V$, the energy of a particle moving on the trajectory $\mathbf{x}(t)$ at time t is defined to be

$$E = \tfrac{1}{2} m|\dot{\mathbf{x}}|^2 + V(\mathbf{x})\ . \tag{37}$$

It then turns out that $\dot{E} = 0$ when the particle moves according to Newton's law $m\ddot{\mathbf{x}} = -\nabla V$ as can be seen by differentiating (37). In this regard, use the Chain rule to write

$$\frac{d}{dt}V(\mathbf{x}(t)) = \nabla V(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) \tag{38}$$

while writing $\frac{d}{dt}(\frac{1}{2}m|\dot{\mathbf{x}}|^2) = m\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}$ and then substituting -∇V for $m\ddot{\mathbf{x}}$ to see that the time derivative of the first term on the right side in (37) just cancels that in the second.

The relation of all of this to (36) is that when E is constant, then the difference on the far right side of (36) is the same as

$$W(g) = -V(\mathbf{x}(b)) + V(\mathbf{x}(a)) . \tag{39}$$

Thus, the work done when a particle moves by Newton's law against a force which is a gradient is given by the difference between the values of the potential function at the end and beginning points of the motion. (We will see in Section 5.2 of the text that this last conclusion is true even if the particle ignores Newton's law.)