Math 21a Supplement on Relativity

Matrices play an interesting role in Einstein's theory of special relativity. To set the stage, imagine two observers with the following characteristics: The first observer is at rest at the origin on the x-axis. This observer measures position along the x-axis with the variable x and the passage of time with a variable t, where $t = 0$ is set to be today at 12 noon here in Boston. Meanwhile, the second observer is moving with constant speed v in the positive x direction, and happens to pass $x = 0$ at $t = 0$. This second observer measures position with a variable x' which takes value zero at his position. Thus, the origin of the x' coordinate system moves in time with respect to that of the x coordinate system, and vice versa. Also, assume that the second observer measures time with a variable t' where $t' = 0$ today at 12 noon in Boston. Finally, assume that these observers had gotten together previously and agreed to measure distance in units of meters and time in units of seconds.

Having set the stage, we now ask the following question:

What is the precise relationship between the (t, x) coordinate and the (t´, x´) coordinates? (1)

To put this in perspective, suppose that the first observer reports a meteor striking the ground at time t and position x, while the second observer reports a meteor strike at time t' and position x' (in his coordinate system). We need to answer the question in (1) in order to decide whether these two observers viewed the same meteor strike.

In the old, Newtonian theory, the relationship between the (t, x) coordinates and the (t', x') coordinates is given by

$$t' = t\ .$$
$$x' = x - vt\ . \tag{2}$$

Note that these last formulas can be written in matrix form by first introducing the column vectors

$$\begin{pmatrix} t \\ x \end{pmatrix} \text{ and } \begin{pmatrix} t' \\ x' \end{pmatrix}. \tag{3}$$

Then, (2) is equivalent to the matrix equation

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}. \tag{4}$$

However, it turns out that our universe <u>does not</u> follow the law in (2) and (4). That is, these transformation rules are demonstrably <u>false</u>. Instead, to all indications, our universe follows the following transformation rules:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = (1 - v^2/c^2)^{-1/2} \begin{pmatrix} 1 & -v/c^2 \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}. \tag{5}$$

Here, c is the speed of light in vacuum, ~ 300,000,000 meters per second. Einstein guessed that (5) should be the rule used by our universe for coordinate transformations. (Even so, the rule in (5) was discovered earlier by Lorentz and is called a <u>Lorentz transformation</u>.)

By the way, when v is any reasonable velocity for the movement of people on earth, then v/c is extremely small. And, in this limit, the transformation in (5) is, for practical purposes, indistinguishable from that in (4).