Math 21a Supplement on Center of Mass

Integrals over volumes occur naturally when studying the motions of extended objects. The fact is that the 'point particle' approximation in Physics is often far from accurate, and in these cases, the spatial extent of an object must be taken into account. In particular, when studying the dynamics of an extended object, three common volume integrals arise; these being

The integral of the mass density to obtain the total mass.
The integral of the position vector to obtain the center of mass.
The integral of the 3×3 matrix **I** whose entry in the i-th column and j-th row is the product of the i-th and j-th components of the position vector. (1)
[One might call this the **moment of inertia matrix**.]

After setting the stage, I shall consider these points in turn. To set the stage, consider an extended object in a volume, V, in $\mathbf{R}^3$. (Here, I will allow for the case V = $\mathbf{R}^3$.) The mass distribution in this object is specified by giving a function, $\sigma(x, y, z)$ on V whose value at any given point specifies the mass density at that point. Thus, for example, σ might be given in the units of kilograms per cubic meter. As an aside, of current intense interest to astronomers is the precise nature of this function σ for our galaxy, or for any other galaxy, for that matter. Any way, as a concrete example for use below, take V = $\mathbf{R}^3$ and $\sigma(\mathbf{r}) = |\mathbf{r}|\,(1 + |\mathbf{r}|^2)^{-5}$.

a) The total mass

Given the mass density function σ, the total mass of the object is given by the volume integral

$$M \equiv \iiint_V \sigma(x, y, z)\, dV \quad . \tag{2}$$

In the specific example where $\sigma(\mathbf{r}) = |\mathbf{r}|/(1 + |\mathbf{r}|^2)^5$, this integral is most easily done as an iterated integral in spherical coordinates. Indeed, introduce the spherical coordinates (ρ, φ, θ) so that

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \,, \\ y &= \rho \sin\phi \sin\theta \,, \\ z &= \rho \cos\phi \,. \end{aligned} \tag{3}$$

Here, $0 \le \rho < \infty$, $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. In these coordinates, $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ and $\sigma = \rho/(1 + \rho^2)^5$. Thus, the integral in (2) is the iterated integral

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \rho^3 (1+\rho^2)^{-5} \sin\phi \, d\rho \, d\phi \, d\theta \tag{4}$$

I will leave it to you (as an exercise) to verify that this integral is equal to $\frac{p}{6}$.
[Hint: The ρ integration yields $\frac{1}{24}\sin\phi$, the subsequent φ integration yields $\frac{1}{24}\,2 = \frac{1}{12}$, and the final θ integration yields $\frac{1}{12}(2\pi) = \frac{p}{6}$.]

b) The center of mass

The center of mass of an extended object in a volume V is the average of the positions of the particles which make up the object. This point in V is the point in $\mathbf{R}^3$ given by the vector

$$\mathbf{X}_{\text{cm}} = \frac{1}{M}\iiint_V \mathbf{r}\, s(\mathbf{r})\, dV \qquad (5)$$

Thus, the x-coordinate of the vector $\mathbf{X}_{\text{cm}}$ is the volume integral over V of the function $M^{-1}\, x\, \sigma(\mathbf{r})$. Likewise, the y-coordinate is the volume integral over V of $M^{-1}\, y\, \sigma(\mathbf{r})$ and the z-coordinate is the volume integral over V of $M^{-1}\, z\, \sigma(\mathbf{r})$.

In our example above, the vector $\mathbf{X}_{\text{cm}}$ is zero and so the center of mass is the origin. To see this, consider, for example the z-coordinate of $\mathbf{X}_{\text{cm}}$. In spherical coordinates, this is the given by the iterated integral

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \rho^4(1+\rho^2)^{-5} \sin\phi\cos\phi\, d\rho\, d\phi\, d\theta\ . \qquad (6)$$

In evaluating this integral, my experience tells me to do the ϕ integration first. The result is

$$\rho^4\,(1 + \rho^2)^{-5}\ \tfrac{1}{2}\ (\sin^2\pi - \sin^2 0) = 0\ . \qquad (7)$$

I'll leave it to you to verify that the x and y components of $\mathbf{X}_{\text{cm}}$ also vanish.
[Hint, do the θ integral first in these cases.]

c) The moment of inertia

The moment of inertia matrix (with respect to the chosen origin of the x-y-z coordinates), $\mathbf{I}$, of our extended body is technically a 3×3 matrix whose entries are as follows: The 1-1 component of I is the integral of $(y^2 + z^2)\,\sigma(x, y, z)$ over the region V. The 1-2 component is the integral of $-xy\,\sigma(x, y, z)$, the 1-3 that of $-xz\,\sigma(x, y, z)$, the 2-1 that of $-yx\,\sigma(x, y, z)$, the 2-2 component is the integral over V of $(x^2 + z^2)\,\sigma(x, y, z)$ and so on. In particular, the entries down the diagonal are integrals over V of $\sigma(x, y, z)$ times $(y^2 + z^2)$, $(x^2 + z^2)$ and $(x^2 + y^2)$, and so the trace of the moment of inertia tensor is the twice the integral of the function $|\mathbf{r}|^2\,\sigma(\mathbf{r})$ over V. That is,

$$\text{trace}\ (\mathbf{I}) = 2\iiint_V (x^2 + y^2 + z^2)\, s(x, y, z)\, dV\ . \qquad (8)$$

In our example, this integral is easiest to compute as an iterated integral in spherical coordinates. Here is the integral:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \rho^5(1+\rho^2)^{-5} \sin\phi\, d\rho\, d\phi\, d\theta \qquad (9)$$

This integral will be left to you to evaluate, but I'll tell you the value: $\frac{p}{6}$. (Hint: The ρ integral gives $\frac{1}{24}\sin\phi$, then the ϕ integral gives $\frac{1}{12}$, and the final θ integral gives 2π times $\frac{1}{12}$.)

By the way, the matrix $\mathbf{I}$ in our example turns out to be diagonal, with each diagonal entry equal to $\frac{p}{18}$.