# Math 21a Handout on Curl and Divergence

Suppose that $\mathbf{F} = (P, Q, R)$ is a vector field on a region in $\mathbf{R}^3$. The text discusses two different ways to differentiate $\mathbf{F}$. The first, the divergence of $\mathbf{F}$, gives a function, while the second, the curl of $\mathbf{F}$, produces another vector field. Here,

$$\text{div } \mathbf{F} = P_x + Q_y + R_z \,,$$

$$\text{curl } \mathbf{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y) \,. \tag{1}$$

There are two tautological, though still important identities that are satisfied by the divergence and the curl, namely

$$\text{curl } \nabla h = \mathbf{0} \,.$$

$$\text{div } (\text{curl } \mathbf{G}) = 0. \tag{2}$$

That is, the curl of a gradient vector field is always zero, and the divergence of a curl is also always zero. (You are asked to prove the latter identity in Problem 9 on page 293.)

Both of the identities in (2) have a converse of sorts:

*For certain kinds of regions in $\mathbf{R}^3$, all vector fields with zero curl are gradients.*

*For certain kinds of regions in $\mathbf{R}^3$, all vector fields with zero divergence are curls.* (3)

Now, it is a bit off the planned path to detail the precise nature of the regions for which each of the conditions in (3) hold. However, it is true that both conditions in (3) hold if the region in question is all of $\mathbf{R}^3$, and if the region is just a ball in $\mathbf{R}^3$. In fact, both conditions hold for any region which is <u>convex</u>, where convex means that the line segment between any two points in the region stays completely in the region. Thus, a cube is convex, but a doughnut is not.

In the case where curl $\mathbf{F} = \mathbf{0}$ in a convex region, a function h whose gradient equals $\mathbf{F}$ can be directly written down: Choose a point, O, in the region as a 'base point'. Then, the function $h$ assigns to any other point X in the region the value of line integral of $\mathbf{F}$ along the line segment from O to X. Indeed, if you declare the point O to be the origin, and think of X as a vector whose tail is at the origin, then the function

$$h(\mathrm{X}) \equiv \int_0^1 (\mathrm{X} \bullet f(t\mathrm{X})) dt \tag{4}$$

has $\nabla h = \mathbf{f}$ if and only if curl $\mathbf{F} = 0$. (It is a real test of your understanding of the meaning of the line integral and the derivative to verify that the gradient of this function gives back the vector field $\mathbf{F}$ if and only if curl $\mathbf{F} = \mathbf{0}$.)

Note that if you study electrostatics, you will find that the electric field in empty space is described by a vector field with zero curl (this is one of Maxwell's equations), and is thus the gradient of a function. The latter is called the 'scalar potential', and it plays an important role in the theory of electric fields.

Meanwhile, if div $\mathbf{F} = 0$, a vector field $\mathbf{G}$ whose curl equals $\mathbf{F}$ is

$$\mathbf{G}(\mathrm{X}) = - \int_0^1 (\mathrm{X} \times f(t\mathrm{X})) t \, dt \,. \tag{5}$$

It is quite a challenge to verify that curl $\mathbf{G} = \mathbf{F}$ if and only if div $\mathbf{F} = 0$

By the way, if you study the physics of magnetism, you will find that the magnetic field in space is described by a vector with zero divergence (this is also one of Maxwell's equations). The fact that the magnetic vector is the curl of another vector (called the vector potential) is quite a useful piece of information in the physics of magnetic fields.