Math 21a Handout on Differential Equations

This handout introduces the subject of differential equations. But it barely scratches the surface of this vast, growing and extremely useful area of mathematics.

1. Differential equations in the sciences

The branch of mathematics called differential equations is a direct application of ideas from calculus, and as this is a mathematics course, I should begin by telling you a little bit about what is meant by the term ‘differential equation’. However, I’ll digress first to begin an argument for including mathematics in the tool kit of even the most experimentally minded scientist.

a) Modeling in the sciences

First, I freely admit to not being an experimentalist. In fact, until recently, I always found the theoretical side of science much more to my liking. Moreover, I suffered from a fairly common misconception:

If I only learn enough mathematics, I can uncover nature’s secrets by pure logical deduction.

I have lately come to the realization that advances in science are ultimately driven by knowledge dug from observations and experiments. Although logic and mathematics can say a great deal about the suite of possible realities, only observation and experimentation can uncover the detailed workings of our particular universe.

With the preceding understood, where is the place for mathematics in an experimentally driven science? The answer to this question necessarily requires an understanding of what modern mathematics is. In this regard, I should say that term ‘mathematics’ covers an extremely broad range of subjects. Even so, a unifying definition might be:

Mathematics consists of the study and development of methods for prediction.

Meanwhile, an experimentally driven science (such as physics or biology or chemistry) has, roughly, the following objective:

To find useful and verifiable descriptions and explanations of phenomena in the natural world.

To be useful, a description need be nothing more than a catalogue or index. But, an explanation is rarely useful without leading to verifiable *predictions*. It is here where mathematics can be a great help. In practice, experimental scientists use mathematics as a tool to facilitate the development of predictive explanations for observed phenomena. And, this is how you can profitably view the role of mathematics. (The use of mathematics as a tool to make predictions of natural phenomena is called modeling and the resulting predictive explanation is often called a mathematical model.)

At this point, it is important to realize that a vast range of mathematics has found applications in the sciences. One, in particular, is differential equations.

b) Equations

The preceding discussion about predictions is completely abstract, and so another digression may prove useful to bring the discussion a bit closer to the earth. In particular, consider what is meant by a prediction: You measure in your lab certain quantities---numbers really. Give these measured quantities letter names such as ‘a’, ‘b’, ‘c’, etc. A prediction can take the form of a formula which determines the value for the quantity c by measuring only the quantities a and b. Such a formula might involve simply an algebraic equation which relates a and b to c.

For example, if you lived in Greece some twenty five hundred years ago, you might discover that the length, c, of the hypotenuse of a right triangle can be predicted from the measured lengths, a and b, of the other two sides. Indeed, if you were Pythagoras, you would write:

$$c = \sqrt{a^2 + b^2}$$

(1.1)

Or, you might determine that the area, A, of a disk can be predicted with knowledge of its radius, r, using the equation

$$A = \pi r^2 .$$

(1.2)

These are examples of algebraic equations in that they involve simple expressions between what is known (a and b in (1.1)) and what is to be predicted (c in (1.1)). A famous and modern algebraic equation is Einstein's formula

$$E = m c^2$$

(1.3)

which describes how the total energy (E) of a body at rest can be computed if you know its mass (m) and the speed of light ($c \approx 3$ million meters per second). A algebraic equation with applications to biology describes how the weight of a body (say w) would change if it had weight w_0 and you hypothetically scaled its length, width and height by the same factor, say s. This formula asserts that

$$w = s^3 w_0 .$$

(1.4)

c) Differential equations.

Differential equations can arise when studying quantities which depend on some auxiliary variable. For example, it is typical in the sciences to study time dependent phenomena. A doctor can be concerned with the amount of a certain medicinal drug in the body as a function of time. That is, there is a function which depends on the variable t = 'time' and its value at time t, say f(t), is the concentration of the medicine at time t in the blood.

Here is another example: An environmental scientist can be concerned with the concentration of mercury in clams along a certain stretch of river. Here, the concentration might depend on the distance downstream. Thus, the concern is with a function which depends on the variable x = 'distance downstream' and its value at distance x, say f(x), is the concentration of mercury in clams which are found at distance x. By the way, this concentration might depend on both position and time--a more complicated situation which shall also concern us.

Here is a third example: A developmental biologist studying fly embryos might be concerned with the level of a certain molecular growth factor as a function of distance from the embryo head. Here, the function in question is the level of the growth factor as a function of the variable which measures the distance from the head of the embryo. Of course, this function can also depend on time as well as position; and it most probably does since live embryos develop as time progresses.

For a fourth example, an epidemiologist might consider the number of deaths from a certain disease as a function of age at death. Here, the variable is the age, α, at death, and the number of deaths of people at age α from the disease gives the function. One could denote the latter by $N(\alpha)$. By the way, this example illustrates an important point: The variable in question need not be time nor a position, but some entirely different quantity. Indeed, the same epidemiologist might consider the average number of heart attack victims in a particular locale, as a function of the level of cholesterol in the victim. In this case, the variable, call it c, is the level of cholesterol, and the function in question assigns to each value of c the number n(c) which is the yearly average of heart attack sufferers with cholesterol level c.

With these examples understood, one can say that a differential equation for a function (or for some collection of functions) of a variable (or collection of variables) is simply an algebraic equation which involves both the function and its *derivatives*. For example, the equation

$$\frac{dp}{dt} = p$$

(1.5)

is a differential equation.

d) Continuity and differentiability

But for a few key exceptions, differential equations involve functions with derivatives, and thus functions which are apriori assumed to be continuous. However, it is important to realize that continuous functions may not be appropriate for describing certain natural processes. For example, the function which assigns to a time t the number N(t) of live bacteria in a petri dish is not a continuous function. Indeed, this function can only take values which are whole numbers, that is N(t) = 0, 1, 2, ... Thus, N(t) is either 3 or 4 but never the number $\pi = 3.1416...$, let alone 3.5.

None the less, it is sometimes a reasonable approximation to reality to pretend that N(t) is continuous. For example, if the number of bacteria is measured in units of 1 million, then N(t) can take on values which differ by .000001; in this case, the approximation that N(t) is continuous might not look so bad. In particular, if the experimental error in counting bacteria is greater than 1, then little is lost by modeling N(t) (here, measured in units of 1 bacteria) as a continuous function. Indeed, in this case, the discreteness of N is effectively invisible. In any event, be forewarned that the use of continuous functions in the sciences constitutes a modeling approximation which may or may not make sense in any given application.

In particular, here is a general rule of thumb which summarizes the preceding discussion:

- *If the true function under discussion jumps in value, then its replacement with a continuous function is reasonable when the experimental error is larger than any of the jumps.*

Of course, one can also consider whether the assumption of differentiability makes sense in any given situation. The rule of thumb here is usually the following:

- *Once the step to a continuous function is made, the step to differentiability rarely adds trauma.*