PROBLEMS:

1. Find, by integration, all solutions to the following differential equatons:

   a) $p' = e^p$.

   b) $p' = e^p + 1$.

   c) $p' = p (1 - p)$.

2. Find 3 real systems (in biology, economics, physics or what ever) where the exponential growth equation $p' = a p$ is a reasonable model. Give a sentence of justification using either the birth/death model in the reading, or via the linear approximation, or through some other mechanism.

3. Find the solution, $u(t, x)$, to

$$\frac{\partial}{\partial t} u(t, x) = -3 \frac{\partial}{\partial x} u(t, x) - 2 u(t, x)$$

   which at $t = 0$ is

   a) e^{-4x}

   b) e^{-x^2}.

4. Consider the solution $u(t, x) = f(x - 3t)$ to the equation $\frac{\partial}{\partial t} u(t, x) = -3 \frac{\partial}{\partial x} u(t, x)$ with the initial condition $u(0, x) = (1 + x^2)^{-1}$. On graph paper, sketch in different colors the graph of the function $y(x)$ in the cases where

   a) $y(x) = u(0, x)$.

   b) $y(x) = u(1, x)$.

   c) $y(x) = u(2, x)$.

5. Let N(t, x) denote the number of people in Boston at time t and age x, both measured in the same units. Explain why this function might be expected to obey an advection equation of the form

$$\frac{\partial}{\partial t}N(t, x) = -\frac{\partial}{\partial x}N(t, x) \; - \; r(x)\, N(t, x).$$

for some suitable function r(x) of age. Indicate in your answer how one should interpret this function r(x) and sketch the r versus x graph of a reasonable possibility. Also, make sure to explain why the speed c in this equation has value +1.

6. Verify by taking the appropriate partial derivatives that $u(t, x) = \frac{a}{t^{1/2}}\, e^{rt}\, e^{-x^2/4\mu t}$ is a solution to

$$\frac{\partial}{\partial t}u = \mu\frac{\partial^2}{\partial x^2}u + r\,u \qquad (*)$$

when μ and r are constants. Here, a is also a constant. (Note that r comes here with a + sign.)

7. Verify that the following functions are also solutions to Equation (*) in the previous problem:

a) $e^{\lambda t}\; e^{x((\lambda - r)/\mu)^{1/2}}$ where λ is any constant with $\lambda > r$.

b) $e^{\lambda t}\; e^{-x((\lambda - r)/\mu)^{1/2}}$ where λ is any constant with $\lambda > r$.

c) $e^{rt}\,(a + bx)$ where a and b are any constants.

d) $e^{\lambda t}\cos[((r - \lambda)/\mu)^{1/2}x]$ where λ is any constant with $\lambda < r$.

e) $e^{\lambda t} \sin[((r - \lambda)/\mu)^{1/2}x]$ where λ is any constant with $\lambda < r$.

8. If a diffusion equation such as that in Equation (*) of Problem 4 is to lead to reasonable predictions of a given phenomena, it is important to discern the appropriate choices for the constants r and μ which appear in the equation. In this regard, it turn out that the values of almost any particular solution u(t, x) to Equation (*) at two fixed times, say t = 1 and t = 2, and two thoughtful choices for x can be used to determine a, r and μ. For example, as you verified in Problem 4, above, the function $u(t, x) = \frac{a}{t^{1/2}} e^{rt} e^{-x^2/4\mu t}$ solves Equation (*); and the values of u(1, x) and u(2, x) for two good choices of x suffice to determine a, r and μ. Indeed, a e^r is given by a = u(1, 0). Choose an additional values for x and then use the resulting values of u at (1, x), (2, 0) and (if necessary) (2, x) to obtain expressions for a, r and μ.

9. This problem concerns the equation $\frac{\partial}{\partial t}u(t, x) = -\frac{\partial}{\partial x}q(t, x) + k(t, x)$. When particle motion is due solely to the ambient fluid moving with velocity, c, I argued that the appropriate choice for q is q = c u.. When particle motion is due solely to random motion, I argued that the right choice for q is $q = -\mu \frac{\partial}{\partial x} u$. What should q be when the particles have both random motion and motion due to the ambient motion of the fluid at velocity c? Give some justification for your choice of q.