Math 21a Supplement on Electricity and Magnetism

Vector fields on $\mathbf{R}^3$ play a central role in Maxwell's theory of electricity and magnetism. In particular, the electric field in space is described in Maxwell's theory by a vector valued function of space and time, $\mathbf{E}(t, x)$. This vector provides the direction and magnitude of the electric field at any given point and at any given time. Likewise, the magnetic field is also described in Maxwell's theory by a vector valued function of space and time, $\mathbf{B}(t, x)$. (Why not the letter '$\mathbf{M}$' for magnetic field? I am not sure. In any event, the traditional letter is '$\mathbf{B}$'.)

Maxwell proposed a set of equations which he postulated constrain the possible vector valued function pairs $(\mathbf{E}, \mathbf{B})$ which can arise as real world electric and magnetic fields. These are the famous Maxwell equations. (Actually, various portions of these equations were written down by others prior to Maxwell's culminating contribution.) These equations involve the operations of curl and divergence in a fundamental way. In any event, here are Maxwell's equations in a vacuum (no charged, polarizable or magnetically susceptible materials or particles present):

$$\text{div } \mathbf{E} = 0,$$
$$\text{div } \mathbf{B} = 0,$$
$$\frac{\partial \mathbf{E}}{\partial t} = \text{curl } \mathbf{B}, \tag{1}$$
$$\frac{\partial \mathbf{B}}{\partial t} = -\text{curl } \mathbf{E} .$$

Here, I have written these equations in units where various natural constants (such as the speed of light) are equal to 1. In a physics book, these equations generally appear with various natural constants whose values depend on the particular choice of units of measurement.

By the way, please take note of the evident symmetric treatment of the electric and magnetic fields here. When there are charged particles present, these equations are modified in a way which breaks the symmetry between $\mathbf{E}$ and $\mathbf{B}$. For example, if charged particles are present, then (1) is modified as follows: First, the distribution of the charged particles in space determines a function, $\rho(t, x)$, which measures the charge density at the point x and at time t. Thus, the triple integral of ρ over any volume V gives the total charge in V at the given time. Second, the particles may be in motion, and a moving charge produces what is called a current. For a single moving charge, this is a vector function of time which is proportional to the velocity vector of the particle. The current due to an ensemble of moving charges (such as electrons moving down a wire) is described by a vector valued function, $\mathbf{J}(t, x)$, which measures the current density at the point x at time t. Thus, the triple integral of a component of $\mathbf{J}$ over a volume V is meant to give the component of the current due to all of the moving charges in V.

Note that the function ρ and the vector valued function $\mathbf{J}$ are not completely independent of each other. Indeed, since the triple integral of ρ measures the total charge at a given time in a region, then this integral will change if charges move in or out of the region. Thus, the change of ρ with respect to time has something to do with the net motion charge across the boundary of the region (in versus out). Meanwhile, moving charges determine the vector valued function $\mathbf{J}$. Thus, we are led to the conclusion that the change of ρ with time must have something to do with $\mathbf{J}$. And, this conclusion is born out by a more careful analysis. In particular, this more careful analysis leads to the following constraint on the possible pairs $(\rho, \mathbf{J})$ which can arise in nature:

$$\frac{\partial \rho}{\partial t} + \text{div } \mathbf{J} = 0 . \tag{2}$$

We shall see in a subsequent supplement how this equation allows one to calculate the time rate of change of charge in a region from the behavior of $\mathbf{J}$ on the region's boundary.

With the function ρ and the vector valued function **J** understood, here are Maxwell's equations in the presence of charges:

$$\text{div } \mathbf{E} = \rho,$$
$$\text{div } \mathbf{B} = 0,$$
$$\frac{\partial \mathbf{E}}{\partial t} = \text{curl } \mathbf{B} - \mathbf{J}, \tag{3}$$
$$\frac{\partial \mathbf{B}}{\partial t} = -\text{curl } \mathbf{E} .$$

Here are some simple examples: First, suppose that a metal ball has total charge q, that the charges are uniformly distributed in a layer near the surface of the ball, and that all of the charges are stationary so that ρ is independent of time and **J** = **0**. If the electric and magnetic fields are assumed to be independent of time also, then the relevant solution to (3) outside the ball (where ρ = 0 too) has the magnetic field **B** = **0**, and the electric field

$$\mathbf{E} = q \ \frac{1}{4\pi|\mathbf{x}|^3} \ \mathbf{x} . \tag{4}$$

Thus, **E** points radially outward from the ball and |**E**| falls off with distance from the center of the ball as the reciprocal of the square of the distance. I leave it to you to verify that div **E** = 0 and also that curl **E** = 0. Note that (4) is not correct inside the ball where the formula for **E** is somewhat more complicated.

For a second example, suppose that electrons are moving at constant speed along a cylindrical wire. Here, lets suppose that the z-axis is the center of the wire, that the wire is electrically neutral, and that the current density, **J**, depends neither on time, nor on the angle in the xy-plane. If the electric and magnetic fields are also assumed to be independent of time, then the relevant solution to (3) outside the wire (where **J** = **0** also) has **E** = **0** and, at a point with coordinates (x, y, z), has

$$\mathbf{B} = \sigma \frac{1}{2\pi(x^2+y^2)} \ (-y, x, 0) , \tag{5}$$

where σ is value of the double integral of the z-component of **J** over the disk cross section of the wire. (The latter disk can be thought of as residing in the xy-plane; it is the intersection of the cylindrical wire with the xy-plane.) I leave it to you to check that div **B** = 0 and also that curl **B** = **0**.

For the third and final example, suppose that there are no charges and no currents, so that (1) is relevant. Here is a solution to (1):

$$\mathbf{E} = \sin(t - z) \ (1, 0, 0) .$$
$$\mathbf{B} = \cos(t - z) \ (0, 1, 0). \tag{6}$$

Note that **E** and **B** both depend on position and time. For example, the places where **E** = **0** move in time. For example, at t = 0, **E** is zero where z = 0, but not so at most later times. Indeed, the places where **E** = **0** are given by

$$z = t + 2\pi n , \tag{7}$$

where n can be any integer. In a very real sense, the electric field in (6) is propagating like a moving wave up the z-axis. And, so is the magnetic field. Such a wave is called an 'electro-magnetic wave'. Radio waves, light waves, gamma rays are all examples of electro-magnetic waves.

Once again, I leave it to you to verify that all of the equations in (1) are obeyed by (6).