Math 21a Supplement on Integration and Electro-Magnetism

The Math 21a Supplement on Electricity and Magnetism presented the equations that describe all known electric and magnetic phenomena. In particular, recall that Maxwell's equations describe electric fields by a vector valued function of space and time, $\mathbf{E}(t, x)$, while magnetic fields are described by a similar object, $\mathbf{B}(t, x)$. Then, Maxwell postulates that only such pairs $(\mathbf{E}, \mathbf{B})$ which obey the constraints

$$\text{div } \mathbf{E} = \rho,$$
$$\text{div } \mathbf{B} = 0,$$
$$\frac{\partial \mathbf{E}}{\partial t} = \text{curl } \mathbf{B} - \mathbf{J}, \tag{1}$$
$$\frac{\partial \mathbf{B}}{\partial t} = -\text{curl } \mathbf{E} .$$

appear in nature. Here, the function ρ and the vector valued function $\mathbf{J}$ (both functions of time and space) are determined by the distribution and velocities of the various charged materials or particles present. The pair $(\rho, \mathbf{j})$ must also satisfy a constraint, which is

$$\frac{\partial \rho}{\partial t} + \text{div } \mathbf{J} = 0 . \tag{2}$$

The purpose of this supplement is illustrate various important ramifications of (1) and (2) which follow from applications of either Stokes' Theorem or the Divergence Theorem.

a) A short review

Before getting to the purpose at hand, I will remind you of two definitions. For the first, suppose that $\mathbf{v}$ is a vector valued function on $\mathbf{R}^3$. (In the examples below the vector valued functions under consideration will be $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{J}$, thus they may also depend on time.) Now, suppose that S is a surface in $\mathbf{R}^3$ with a chosen normal direction, $\mathbf{n}$. For example, S could be the whole surface of a sphere and $\mathbf{n}$ the outward pointing normal, or S could be something much more intricate. With regard to the normal vector $\mathbf{n}$, keep in mind that most surfaces have two possible normal directions which point opposite.

The first definition to review is that of the <u>flux</u> of $\mathbf{v}$ through S in the direction of the normal vector $\mathbf{n}$. In particular, remember that this flux is defined to be the value of the surface integral

$$\int_S \mathbf{v} \cdot \mathbf{n} \, dS . \tag{3}$$

Why call (3) 'flux'? Here is one reason: The value of (3) is evidently proportional to the average amount that $\mathbf{v}$ points outward from S along the normal $\mathbf{n}$. Thus, if $\mathbf{v}$ is the velocity vector of a moving fluid at each point, then the positivity of (3) indicates that there is a net flow of fluid through S in the direction $\mathbf{n}$. On the other hand , if (3) is negative, then the net flow of fluid through S is in the direction of $-\mathbf{n}$. So, when $\mathbf{v}$ is the velocity of a fluid, the integral in (3) acts just like flux should act. When $\mathbf{v}$ represents some other physical quantity, the corresponding integral in (3) is still called a flux integral.

The second definition in this review requires the specification of a closed path γ in $\mathbf{R}^3$. That is, γ is some (possibly very twisty) path whose endpoint coincides with its starting point. Having chosen a direction around γ, we introduced the path integral

$$\int_\gamma \mathbf{v} \cdot d\mathbf{x} . \tag{4}$$

In the physics literature, this integral is called the <u>circulation</u> of **v** around γ. (If **v** is supposed to represent the velocity vector of a moving fluid, then positivity of (4) indicates that the fluid is somewhat whirlpool-like and with a net rotation around γ. On the other hand if (4) is negative, then the fluid should also be somewhat whirlpool-like, but rotating in the opposite sense to which γ is traversed. This is, I believe, the historical basis for using the term 'circulation' for the integral in (4).)

b) Charges, currents and flux

With the preceding definitions understood, return now to Equations (1) and (2). In particular, consider first (2). The function ρ in (2) is supposed to give the density of electric charges at each point in space and at each time. This means no more nor less then the assertion that the integral over any volume V of ρ gives the total charge in V. Let $Q_V(t)$ denote this total charge. Since ρ is a function of t, so is the charge in V. In particular, $Q_V(t)$ can change with time if there is a net movement of charge into or out of V. Since the current density **J** describes moving charges, there should be some relationship between the time derivative of $Q_V(t)$ and the flux of **J** across the boundary surface S of the volume V. And, with the help of the Divergence Theorem, (2) predicts just such a relationship. To see how this comes about, first invoke the definition of Q_V as the integral of ρ over V to write

$$\frac{\partial Q_V}{\partial t} = \iiint_V \frac{d\rho}{dt}\, dV \ . \tag{5}$$

Next, invoke (2), to rewrite the right hand side above as

$$\iiint_V \frac{d\rho}{dt}\, dV = -\iiint_V \text{div } \mathbf{J}\, dV \ . \tag{6}$$

Finally, invoke the divergence theorem to rewrite the right hand side of (6) as

$$-\iiint_V \text{div } \mathbf{J}\, dV = -\iint_S \mathbf{J}\cdot\mathbf{n}\, dS \ , \tag{7}$$

where **n** is the outward pointing normal vector to S. Together, these manipulations of the right hand side of (5) produce the equality

$$\frac{\partial Q_V}{\partial t} = -\iint_S \mathbf{J}\cdot\mathbf{n}\, dS \ . \tag{8}$$

This equation asserts that the rate of change of the total charge in V is minus the outward flux of current through the boundary of V.

c) Electric fluxes

Return now to Equation (1). In particular, the Divergence Theorem can be used with the first line in (1) to equate the total charge in V, our Q_V, with the flux of the electric field **E** through the surface. Indeed, integrate both sides of (1) over the volume V. The integral of the right hand side gives Q_V by definition. Meanwhile, according to the Divergence Theorem,

$$\iiint_V \text{div } \mathbf{E}\, dV = \iint_S \mathbf{E}\cdot\mathbf{n}\, dS \ . \tag{9}$$

Thus, the first of the Maxwell equations implies that

$$Q_V(t) = \iint_S \mathbf{E}\cdot\mathbf{n}\, dS \ . \tag{10}$$

Here is an example: In the previous supplement, I asserted that $\mathbf{E} = q\,(4\pi)^{-1}\,|\mathbf{x}|^{-3}\,\mathbf{x}$ and $\mathbf{B} = \mathbf{0}$ is a solution to (1) outside of a ball centered at the origin. Here, q can be any constant. I also stated that q can be interpreted as the net electric charge in the ball. This interpretation of the constant q can be verified by computing the integral on the right hand side of (10) in the case where S is a sphere of radius sufficiently large to enclose the ball of charge. Indeed, take R to be such a radius. The outward pointing normal vector $\mathbf{n}$ to such a sphere is $\mathbf{n} = \dfrac{\mathbf{x}}{|\mathbf{x}|}$, and so

$$\mathbf{E}\bullet\mathbf{n} = q\,(4\pi)^{-1}\,|\mathbf{x}|^{-4}\,\mathbf{x}\cdot\mathbf{x} = q\,(4\pi)^{-1}R^{-2} \tag{11}$$

since $\mathbf{x}\cdot\mathbf{x} = |\mathbf{x}|^2$ and $|\mathbf{x}|^2 = R^2$ is the equation for the sphere S. Thus, the integrand in (10) is constant in this case, and so the value of (10) is equal to this constant integrand times the area of the sphere. The latter is $4\pi R^2$, so the value for Q_V in (10) is, indeed, the number q.

By the way, note that in general, (10) expresses $Q_V(t)$ in terms of the electric flux through a surface S, while (8) expresses the time derivative of $Q_V(t)$ in terms of the flux of the current through S. Are these two expressions always compatible? They are indeed, and to see why, start by taking the time derivative of both sides of (10) to obtain

$$\frac{\partial Q_V}{\partial t} = \iint_S \frac{\partial \mathbf{E}}{\partial t}\bullet\mathbf{n}\; dS\;. \tag{12}$$

Now, use the third line in (1) to rewrite the time derivative of $\mathbf{E}$ in (12) so that the result is

$$\frac{\partial Q_V}{\partial t} = \iint_S (\mathrm{curl}\,\mathbf{B})\bullet\mathbf{n}\; dS \; - \; \iint_S \mathbf{J}\bullet\mathbf{n}\; dS\;. \tag{13}$$

Now, this is exactly the same as (8) provided that

$$\iint_S (\mathrm{curl}\,\mathbf{B})\bullet\mathbf{n}\; dS \; = 0 \tag{14}$$

whenever S is a surface which completely encloses a volume V. This last conclusion is correct, and it follows with the help of the Divergence Theorem. Indeed, first apply the Divergence Theorem in reverse rewrites the left hand side of (14) as

$$\iint_S (\mathrm{curl}\,\mathbf{B})\bullet\mathbf{n}\; dS \; = \iiint_V \mathrm{div}\,(\mathrm{curl}\,\mathbf{B})\; dV\;. \tag{15}$$

Then, recall that the divergence of any curl is automatically zero. That is, div (curl $\mathbf{v}$) = 0 for any choice of vector-valued function $\mathbf{v}$. (If you don't remember this, you can easily reprove it using the definitions of divergence and curl with the fact that when taking second derivatives in differing directions, the order of which direction is first and which is second is irrelevant.)

d) Magnetic fluxes

Where as the electric flux through a surface enclosing a volume gives the total charge in the volume, the magnetic flux through such a surface is always zero. This is a consequence of the second equation in (1). To see how, first integrate both sides of this equation 0 = div $\mathbf{B}$ over a volume V. Then, use the Divergence Theorem to rewrite the resulting equality as

$$0 = \iiint_V \mathrm{div}\,\mathbf{B}\; dV \; = \iint_S \mathbf{B}\bullet\mathbf{n}\; dS\;. \tag{16}$$

You might ask whether this last equation is consistent with the last equation in (1). It certainly is, as I'll demonstrate: Differentiate both sides of the equality $0 = \iiint_V \text{div } \mathbf{B} \; dV$ with respect to time to find that

$$0 = \iiint_V \text{div } \left(\frac{\partial \mathbf{B}}{\partial t}\right) dV . \tag{17}$$

Now, substitute -curl $\mathbf{E}$ for the time derivative of $\mathbf{B}$ and conclude that the right hand side of (17) is zero by using the fact that div (curl $\mathbf{E}$) = 0.

e) Circulation

Suppose not that γ is a closed path in space. Here, γ could be quite twisty, but its starting point should be the same as its ending point. Also, choose a direction to traverse γ since of interest to us here is the circulation of the magnetic field around γ. Suppose as well that S is a surface in space whose boundary is the curve γ.

According to Stokes' theorem,

$$\int_\gamma \mathbf{B} \cdot d\mathbf{x} = \iint_S (\text{curl } \mathbf{B}) \cdot \mathbf{n} \; dS , \tag{18}$$

where $\mathbf{n}$ is a suitably chosen normal vector to S. Now, the third of the Maxwell equations in (1) can be used to rewrite the surface integral on the right side of (18) in terms of the electric field and the current. In particular, this rewriting turns (18) into

$$\int_\gamma \mathbf{B} \cdot d\mathbf{x} = \iint_S \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{n} \; dS + \iint_S \mathbf{J} \cdot \mathbf{n} \; dS . \tag{19}$$

In particular, this equation asserts that when the electric field is time independent, then the circulation of $\mathbf{B}$ around γ is equal to the flux of the current through any surface whose boundary is equal to γ.

For an example, consider the solution to the Maxwell equation from the previous handout which I claimed gives time independent electric and magnetic fields outside of a vertical, cylindrical wire with a suitably uniform current $\mathbf{J}$ inside and pointed along the z-axis. This solution has $\mathbf{E} = 0$ and it has $\mathbf{B} = \sigma \, (2\pi \, (x^2 + y^2))^{-1} \, (-y, x, 0)$ at the point (x, y, z). In the aforementioned handout, I claimed that the constant σ was the flux of the current $\mathbf{J}$ across the disk in which the wire intersects the xy-plane.

Equation (19) can be used to verify this assertion as follows: First, choose some large radius R so that the circle in the xy-plane with this radius lies outside the wire and surrounds it. This circle is going to be the path γ for the left hand side of (19). Parameterize γ by sending $t \to \mathbf{x}(t) = (\text{R} \cos t, \text{R} \sin t, 0)$ for $0 \le t \le 2\pi$. Given this parameterization, the line integral on the left hand side of (19) is

$$\int_\gamma \mathbf{B} \cdot d\mathbf{x} = \int_0^{2\pi} \mathbf{B}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \; dt , \tag{20}$$

where $\mathbf{x}'(t) = (-\text{R} \sin t, \text{R} \cos t, 0)$ is the time derivative of $\mathbf{x}(t)$. Now, to compute the integrand in (20), the first observation to is that $\mathbf{B}(\mathbf{x}(t)) = \sigma \, (2\pi \, \text{R}^2)^{-1} \, (-\text{R} \sin t, \text{R} \cos t, 0)$. Thus, $\mathbf{B}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = \sigma \, (2\pi \, \text{R}^2)^{-1} \, \text{R}^2 = \sigma/(2\pi)$. As this $\sigma/(2\pi)$ is independent of time, the integral on the right side of (20) simply multiplies $\sigma/(2\pi)$ by 2π. Thus, the circulation of $\mathbf{B}$ around this loop is given by σ.

Meanwhile, take the surface S on the right hand side of (19) to be the disk in the xy-plane whose boundary is the circle of radius R. Then, (19) asserts that σ is the flux of $\mathbf{J}$ through this disk. As there is no flux outside of the wire, this is the same as the flux of $\mathbf{J}$ through the disk which is given by the intersection of the wire with the xy-plane.