Math 21a Handout on Surface Area

The text argued that after parameterizing a surface (or part of one) by a function $\mathbf{X}(u, v)$, with u and v coordinates on a region R in $\mathbf{R}^2$, then the integral of a function f on the surface is the same as the integral

$$\iint_S f\,dS \equiv \iint_R f(\mathbf{X}(u,v))\,|\mathbf{X}_u \times \mathbf{X}_v|\,du dv \tag{1}$$

The purpose of this handout is to provide three examples of the preceding formula, but all with the same surface and same function f. Rather, each example will use a <u>different</u> parameterization of the surface, and give evidence for the (not obvious) fact that the integral in (1) is truly independent of your chosen way of parameterizing the surface. In particular, the surface in question is the top half of the unit sphere in $\mathbf{R}^3$, that is where

$$\begin{gathered} x^2 + y^2 + z^2 = 1, \\ z \geq 0. \end{gathered} \tag{2}$$

Meanwhile, the function f is just the z-coordinate in these examples.

<u>Example 1</u>: This example computes the integral in 1 using the parameterization where

$$\mathbf{X}(u, v) = (u, v, (1 - u^2 - v^2)^{1/2}) . \tag{3}$$

where (u, v) are restricted to the disk D where $u^2 + v^2 \leq 1$.

In this case,

$$\begin{gathered} \mathbf{X}_u = (1, 0, -u/(1 - u^2 - v^2)^{1/2}) , \\ \mathbf{X}_v = (0, 1, -v/(1 - u^2 - v^2)^{1/2}) , \\ \mathbf{X}_u \times \mathbf{X}_v = (u/(1 - u^2 - v^2)^{1/2}, v/(1 - u^2 - v^2)^{1/2}, 1) \\ |\mathbf{X}_u \times \mathbf{X}_v| = 1/(1 - u^2 - v^2)^{1/2} . \end{gathered} \tag{4}$$

Meanwhile, $z = (1 - u^2 - v^2)^{1/2}$, so the integrand $z\,|\mathbf{X}_u \times \mathbf{X}_v|$ in (1) is just the constant 1. Thus, the integral in 1 is simply the area of the unit radius disk which is π.

<u>Example 2</u>: In this example, the top half of sphere is parameterized as

$$\mathbf{X}(u, v) = ((1 - u^2)^{1/2} \cos v, (1 - u^2)^{1/2} \sin v, u) , \tag{5}$$

where $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$. (You should convince yourself that the expression in (5) really parameterizes the top half of the sphere.) In this case,

$$\begin{gathered} \mathbf{X}_u = (- u (1 - u^2)^{-1/2} \cos v, -u (1 - u^2)^{-1/2} \sin v, 1), \\ \mathbf{X}_v = (-(1 - u^2)^{1/2} \sin v, (1 - u^2)^{1/2} \cos v, 0) , \\ \mathbf{X}_u \times \mathbf{X}_v = (-(1 - u^2)^{1/2} \cos v, -(-1 - u^2)^{1/2} \sin v, -u) \\ |\mathbf{X}_u \times \mathbf{X}_v| = 1. \end{gathered} \tag{6}$$

Thus, in this parameterization, the integrand in (1) is z $|\mathbf{X}_u \times \mathbf{X}_v| = u$, and the integral in (1) is

$$\int_0^{2\pi} \int_0^1 u \, du \, dv \ , \tag{7}$$

which is equal to π also.

Example 3: In this example, the top half of the sphere is parameterized using spherical coordinates, so

$$\mathbf{X}(u, v) = (\cos v \sin u, \sin v \sin u, \cos u) \ , \tag{8}$$

where $0 \le u \le \pi/2$ and $0 \le v \le 2\pi$. Here,

$$\begin{gathered} \mathbf{X}_u = (\cos v \cos u, \sin v \cos u, -\sin u) \ . \\ \mathbf{X}_v = (-\sin v \sin u, \cos v \sin u, 0) \ , \\ \mathbf{X}_u \times \mathbf{X}_v = \sin u \, (\cos v \sin u, \sin v \sin u, \cos u) \ , \\ |\mathbf{X}_u \times \mathbf{X}_v| = \sin u \ . \end{gathered} \tag{9}$$

Thus, the integrand in (1) is $z \, |\mathbf{X}_u \times \mathbf{X}_v| = \cos u \sin u = \frac{1}{2} \sin 2u$ and the integral in (1) is

$$\tfrac{1}{2} \int_0^{2\pi} \int_0^{\pi/2} \sin(2u) du dv \tag{10}$$

Here, the u integral gives a factor of $\frac{1}{2}$ $(\cos 2u|_0 - \cos 2u|_{\pi/2})$, which is 1, while the v integral gives a factor of 2π and then the factor of $\frac{1}{2}$ out in front makes the whole expression in (10) equal to π as it should be.