Math 21a Handout on Triple Integrals

The purpose of this handout is to provide a few more examples of triple integrals. In particular, we provide one example in the usual x,y,z coordinates, one in cylindrical coordinates and one in spherical coordinates.

<u>Example 1</u>: Integrate the function $f(x, y, z) = z$ over the tetrahedral pyramid in space where

$$\begin{gathered} 0 \leq x. \\ 0 \leq y. \\ 0 \leq z. \\ x + y + z \leq 1 \end{gathered} \tag{1}$$

The integral in question is

$$\mathrm{I} \equiv \iint_R \left(\int_0^{1-x-y} z\, dz \right) dA \tag{2}$$

where R is the 'shadow' region in the xy-plane; the region where

$$\begin{gathered} 0 \leq x \\ 0 \leq y \\ x + y \leq 1 \end{gathered} \tag{3}$$

Indeed, a vertical line (where x and y are constant) will hit the pyramid only if x and y are non-negative and $x + y$ 1. Otherwise, one of the conditions in (1) is violated when $z \geq 0$. This line enters our pyramid from below where $z = 0$ and it exits with z value where $x + y + z = 1$, which is to say where $z = 1 - x - y$. This information provides the lower bound to the z-integral, 0, and also the upper bound, $1 - x - y$.

The integral over the z coordinate in (2) gives $\frac{1}{2}(1 - x - y)^2$, and so

$$\mathrm{I} = \tfrac{1}{2} \int_0^1 \left(\int_0^{1-x} (1 - x - y)^2\, dy \right) dx \tag{4}$$

With regard to the upper and lower bounds to the x and y integrals in (4), remark that a vertical line (where x is constant) hits the shadow region R only if $0 \leq x \leq 1$. Otherwise, one of the conditions in (3) will be violated when y is non-negative. (These bounds on x give the lower and upper bounds for the x integral in (4).) This line enters the shadow region where $x = 0$ and exits with y value where $x + y = 1$. In this way, the lower bound to the y integral in (4) is found to be 0 and the upper bound is found to be $1 - x$.

In any event, the y integral in (4) gives $\frac{1}{3}(1 - x)^3$ which leaves

$$\mathrm{I} = \tfrac{1}{6} \int_0^1 (1 - x)^3\, dx = \tfrac{1}{24}. \tag{5}$$

<u>Example 2</u>: The problem in this example is to integrate the function z over the region where

$$\begin{gathered} 0 \leq z \\ x^2 + y^2 \leq 1 \\ z \leq 1 - x^2 - y^2 \end{gathered} \tag{6}$$

To accomplish this task, note that the integration region as just described involves only the coordinates x and y through the combination $x^2 + y^2$, and the function to be integrated doesn't involve these coordinates at all. In particular, since x and y only appear here through $x^2 + y^2$, it makes sense to solve the problem using the cylindrical coordinates (r, θ, z). In terms of these coordinates, the integral in question has the form

$$\mathrm{I} = \iint_{r\leq 1}\left(\int_0^{1-r^2} z\,dz\right) r\,dr\,dq \tag{7}$$

Here, the shadow region is seen to be the disk in the xy-plane where $r \leq 1$ since a vertical line which hits the xy-plane outside of this disk violates the middle point in (6). Meanwhile, the lower and upper bounds for the z-integral in (7) come about via the observation that a vertical line which has $r \leq 1$ enters the region where $z = 0$ (due to the first bound in (6)) and exits where $z = 1 - r^2$ (due to the third bound in (6)).

In any event, the z-integral in (7) gives $\frac{1}{2}(1 - r^2)^2$ which makes

$$\mathrm{I} = \tfrac{1}{2}\int_0^1 (1-r^2)^2\left(\int_0^{2\pi} dq\right) r\,dr \tag{8}$$

Here, the bounds of 0 and 1 for the r integral come from the fact that the shadow region is the disk in the xy-plane where $r \leq 1$. This fact also explains the 0 and 2π bounds for the θ integral.

The θ integral in (8) gives 2π, which implies that

$$\mathrm{I} = \pi\int_0^1 (1-r^2)^2\, r\,dr \tag{9}$$

This last integral can be done by substituting $u = r^2$. The answer is $\mathrm{I} = \frac{p}{6}$.

<u>Example 3</u>: The problem here is to integrate the function $f(x, y, z) = z$ over the upper half of the ball; this being the region where

$$\begin{gathered} 0 \leq z. \\ x^2 + y^2 + z^2 \leq 1 \end{gathered} \tag{10}$$

Although this problem can be worked in cylindrical coordinates, spherical coordinates work as well. For the latter, the iterated integral is

$$\mathrm{I} = \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 (r\cos f)\, r^2 \sin f\, dr\; dq\; df \tag{11}$$

Concerning (11), some explanation is in order: First, the term ρ cos ϕ in the parenthesis is just the expression for z in spherical coordinates. Second, the integration bounds of 0 and 1 for ρ, 0 and 2π for θ, and 0 and π/2 for ϕ insure that the integration takes place only over the upper half of the ball. In particular, the function z is non-negative only where ϕ ≤ π/2 and so the π/2 upper bound on the ϕ integral restricts the integration as required.

The ρ integration in (11) can be done first with the result being $\frac{1}{4}$. The θ integration then provides a factor of 2π. Finally, the ϕ integration provides a factor of $\frac{1}{2}$. (Use the substitution $u = \cos\phi$ while noticing that du = -sin ϕ dϕ.) Thus, I = π/4.

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