

Math 21a Handout on Triple Integrals

The purpose of this handout is to provide a few more examples of triple integrals. In particular, I provide one example in the usual x-y-z coordinates, one in cylindrical coordinates and one in spherical coordinates.

Example 1: Here is the problem: Integrate the function $f(x, y, z) = z$ over the tetrahedral pyramid in space where

- $0 \leq x$.
 - $0 \leq y$.
 - $0 \leq z$.
 - $x + y + z \leq 1$.
- (1)

The integral in question is

$$I \equiv \iint_R \left(\int_0^{1-x-y} z dz \right) dA ,$$

(2)

where R is the ‘shadow’ region in the x-y plane; the region where

- $0 \leq x$,
 - $0 \leq y$,
 - $x + y \leq 1$.
- (3)

Indeed, a vertical line (where x and y are constant) will hit the pyramid only if x and y are non-negative and $x + y \leq 1$. Otherwise, one of the conditions in (1) is violated when $z \geq 0$. This line enters our pyramid from below where $z = 0$ and it then exits with z value where $x + y + z = 1$, which is to say where $z = 1 - x - y$. This information provides the lower bound to the z-integral, 0, and also the upper bound, $1 - x - y$.

The integral over the z coordinate in (2) gives $2^{-1} (1 - x - y)^2$, and so

$$I = 2^{-1} \int_0^1 \left(\int_0^{1-x} (1 - x - y)^2 dy \right) dx .$$

(4)

With regard to the upper and lower bounds to the x and y integrals in (4), remark that a vertical line (where x is constant) hits the shadow region R only if $0 \leq x \leq 1$. Otherwise, one of the conditions in (3) will be violated when y is non-negative. (These bounds on x give the lower and upper bounds for the x integral in (4).) This line enters the shadow region where $x = 0$ and exits with y

value where $x + y = 1$. In this way, the lower bound to the y integral in (4) is found to be 0 and the upper bound is found to be $1 - x$.

In any event, the y integral in (4) gives $3^{-1} (1 - x)^3$ which leaves

$$I = 6^{-1} \int_0^1 (1 - x)^3 dx = (24)^{-1} . \tag{5}$$

Example 2: The problem in this example is to integrate the function z over the region where

- $0 \leq z,$
 - $x^2 + y^2 \leq 1,$
 - $z \leq 1 - x^2 - y^2 .$
- (6)

To accomplish this task, note that the integration region as just described involves only the coordinates x and y through the combination $x^2 + y^2$, and the function to be integrated doesn't involve these coordinates at all. In particular, since x and y only appear here through $x^2 + y^2$, it makes sense to solve the problem using the cylindrical coordinates (r, θ, z) . And, in terms of these coordinates, the integral in question has the form

$$I = \iint_{r \leq 1} \left(\int_0^{1-r^2} z dz \right) r dr d\theta . \tag{7}$$

Here, the shadow region is seen to be the disk in the x - y plane where $r \leq 1$ since a vertical line which hits the x - y plane outside of this disk violates the middle point in (6). Meanwhile, the lower and upper bounds for the z -integral in (7) come about via the observation that a vertical line which has $r \leq 1$ enters the region where $z = 0$ (because of the first point in (6)) and exits where $z = 1 - r^2$ (because of the third point in (6)).

In any event, the z -integral in (7) gives $2^{-1} (1 - r^2)^2$ which makes

$$I = 2^{-1} \int_0^1 (1 - r^2)^2 \left(\int_0^{2\pi} d\theta \right) r dr \tag{8}$$

Here, the bounds of 0 and 1 for the r integral come from the fact that the shadow region is the disk in the x - y plane where $r \leq 1$. This fact also explains why the 0 and 2π bounds for the θ integral.

The θ integral in (8) gives 2π , which implies that

$$I = \pi \int_0^1 (1 - r^2)^2 r dr . \tag{9}$$

This last integral can be done by substituting $u = r^2$. The answer is $I = \pi/6$.

Example 3: The problem here is to integrate the function $f(x, y, z) = z$ over the upper half of the ball; this being the region where

- $0 \leq z$.
 - $x^2 + y^2 + z^2 \leq 1$.
- (10)

Although this problem can be worked in cylindrical coordinates, spherical coordinates work as well. For the latter, the iterated integral is

$$I = \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi .$$

(11)

Concerning (11), some explanation is in order: First, the term $\rho \cos \phi$ in the parenthesis is just the expression for z in spherical coordinates. Second, the integration bounds of 0 and 1 for ρ , 0 and 2π for θ and 0 and $\pi/2$ for ϕ insure that the integration takes place only over the upper half of the ball. In particular, the function z is non-negative only where $\phi \leq \pi/2$ and so the $\pi/2$ upper bound on the ϕ integral restricts the integration as required.

The ρ integration in (11) can be done first with the result being $1/4$. The θ integration then provides a factor of 2π . Finally, the ϕ integration provides a factor of $1/2$. (Use the substitution $u = \cos \phi$ while noticing that $du = -\sin \phi \, d\phi$.) Thus, $I = \pi/4$.