9 Sample Exam

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. (a) Consider the two surfaces $\rho = 3\csc\phi$ (given in spherical coordinates) and $r = 3$ (given in cylindrical coordinates). Are they the same surface, or are they different surfaces? Explain your answer.

(b) Consider the two surfaces $\sin\phi = \cos\phi$ (given in spherical coordinates) and $z = \sqrt{r^2}$ (given in cylindrical coordinates). Are they the same surface, or are they different surfaces? Explain your answer.

2. Describe the following in rectangular coordinates:

(a) The intersection of the surfaces given in cylindrical coordinates by $\theta = \frac{\pi}{4}$ and $\theta = \frac{2\pi}{3}$

(b) The intersection of the surfaces given in spherical coordinates by $\rho = 1$ and $\theta = \frac{\pi}{2}$

3. Find a pair of values x, y with $0 < x < 1, 0 < y < 1$ such that $f(x, y) \geq 8$, or show that no such values x and y can exist, for each of the following functions of two variables:

(a) $f(x,y) = \dfrac{1}{x} + \dfrac{1}{y}$

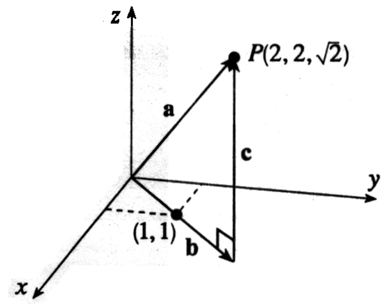
(b) $f(x,y) = 2^{xy+2.5}$

(c) $f(x,y) = \dfrac{1}{x^2 + y^2 + 1}$

(d) $f(x,y) = \cos\left(2x + 3y + \ln\left(x^2 + y^2\right)\right)$

(e) $f(x,y) = \dfrac{1}{x - y + 1}$

4. Let $\mathbf{a} = \overrightarrow{OP}$, where P is the point $\left(2, 2, \sqrt{2}\right)$. Compute the vectors $\mathbf{b}$ and $\mathbf{c}$.



5. Let $\mathbf{a} = (x + y)\,\mathbf{i} + 2\mathbf{j} + y\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + (4x + y + 1)\,\mathbf{j} + 4\mathbf{k}$.

(a) Find values of x and y such that $\mathbf{a} \perp \mathbf{b}$.

(b) Find values of x and y such that $\mathbf{a} \parallel \mathbf{b}$. (*Hint:* Assume that $c\mathbf{a} = \mathbf{b}$ for some value c.)

6. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be three vectors in the plane $3x - 5y + 6z = 7$. Compute $(-\mathbf{a} + 4\mathbf{b} - 7\mathbf{c}) \cdot (-3\mathbf{i} + 5\mathbf{j} - 6\mathbf{k})$.

7. Let $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$ be distinct non-zero vectors in space. Which of the following must be true, which might be true, and which cannot be true? Justify your answers.

(a) If $\mathbf{r} \parallel \mathbf{s}$ and $\mathbf{s} \parallel \mathbf{t}$, then $\mathbf{r} \parallel \mathbf{t}$.

(b) If $\mathbf{r} \perp \mathbf{s}$ and $\mathbf{s} \perp \mathbf{t}$, then $\mathbf{r} \perp \mathbf{t}$.

(c) If $\mathbf{r} \times (\mathbf{s} \times \mathbf{t}) = \mathbf{0}$ and $\mathbf{s} \times \mathbf{t} \neq \mathbf{0}$, then $\mathbf{r} \perp (\mathbf{s} + \mathbf{t})$.

(d) If $\mathbf{r} \cdot (\mathbf{s} \times \mathbf{t}) = 0$ and $\mathbf{s} \times \mathbf{t} \neq \mathbf{0}$, then $\mathbf{r} \perp (\mathbf{s} + \mathbf{t})$.

8. Suppose we have three distinct unit vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ which satisfy the following conditions:

(i) $\mathbf{b} \times \mathbf{c} \neq \mathbf{0}$ (ii) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{0}$

Which of the following must be true, which might be true, and which cannot be true? Justify your answers.

(a) $\mathbf{b}$ is perpendicular to $\mathbf{c}$.

(b) $\mathbf{a}$ is perpendicular to $\mathbf{c}$.

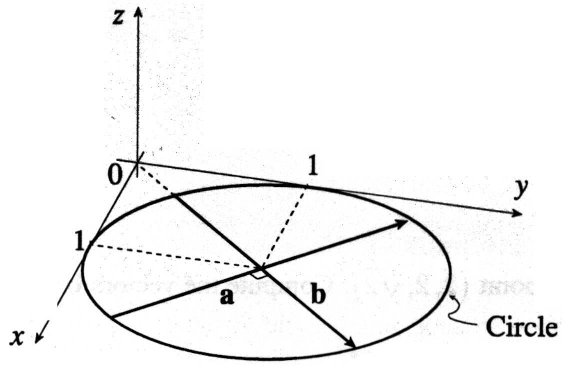
(c) $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

9. Describe and sketch the surfaces in space defined by the following equations.

(a) $y = -z + 1$

(b) $x^2 + y^2 = 3$

10. Referring to the diagram below, give the component representation of each vector.



(a) $\mathbf{a}$

(b) $\mathbf{b}$

(c) $\mathbf{a} \times \mathbf{b}$

(d) $\mathbf{a} + \mathbf{b}$

(e) $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} \times \mathbf{b})$

11. Let $\mathbf{N} = \mathbf{i} - \mathbf{j} + \mathbf{k}$.

(a) What is the equation of the plane P containing $(0, 0, 0)$ with normal vector $\mathbf{N}$?

(b) Find two unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in the plane P which are not parallel to one another.

(c) What is the relationship between $\mathbf{N}$ and $\mathbf{u}_1 \times \mathbf{u}_2$?

12. (a) Show that the line given by $x = t$, $y = 3t - 2$, $z = -t$ intersects the plane $x + y + z = 1$

(b) Find a point of intersection.

13. Consider the plane $x + y + z = 0$.

(a) Give three distinct points with integer coordinates that lie on this plane.

(b) Find the area of the triangle formed by those three points.

14. A particle moves in such a way that its path traces out the circle $x^2 + y^2 = 4$, z 3.

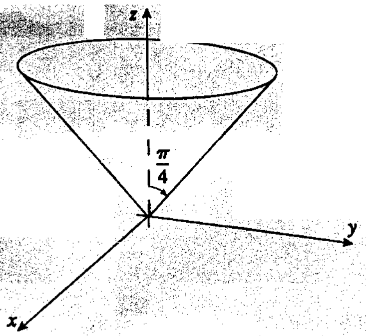
(a) Write an equation of the curve traced out by the particle in cylindrical coordinates.

(b) Write an equation of the curve traced out by the particle in spherical coordinates.

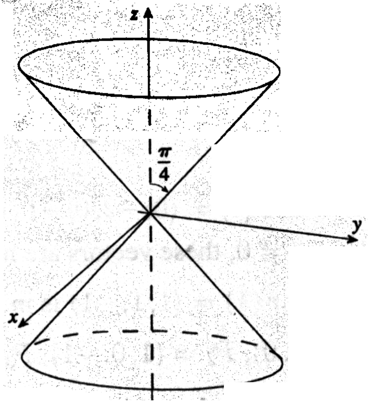
9 Sample Exam Solutions

1. (a) $\rho = 3\csc\phi \;\Rightarrow\; \rho\sin\phi = 3$ or r 3 in cylindrical coordinates. These are the same surface, a cylinder of radius 3.

(b) The surfaces are different. The surface $\cos\phi = \frac{1}{\sqrt{2}}$, $\rho \geq 0$ is a single cone, and the surface z r is a double cone.



$\cos\phi = \frac{1}{\sqrt{2}}, \rho \geq 0$



$z = r$

2. (a) The intersection of two half-planes, namely, the z-axis

(b) The circle $y^2 + z^2$ 1 in the yz-plane (x

3. (a) $x = 0.25$, $y = 0.25$ gives $f(x, y) = 8$.

(b) $x = 0.9$, $y = 0.9$ gives $f(x, y) \approx 9.9$.

(c) $\frac{1}{2} \leq \frac{1}{x^2 + y^2 + 1} \leq 1$

(d) $|\cos w| \leq 1$ for any w

(e) $x = 0.09$, $y = 0.99$ gives $f(x, y) = 10$

4. $\mathbf{b} = \langle 2, 2, 0 \rangle$, $\mathbf{c} = \langle 0, 0, \sqrt{2} \rangle$

5. (a) $x = -1$, $y = 1$ (among others) (b) x -2, $y = 8$

6. 0

7. (a) True

(b) Might be true; false if $\mathbf{r} = \mathbf{t}$

(c) True: $\mathbf{r}$ is parallel to $\mathbf{s} \times \mathbf{t}$

(d) False: $\mathbf{r} = \mathbf{s} + \mathbf{t} \perp \mathbf{s} \times \mathbf{t}$, but $\mathbf{r} \cdot (\mathbf{s} + \mathbf{t}) \neq 0$

8. (a) Might be true

(b) True

(c) True

9. (a) $y + z = 1$, a plane

(b) $x^2 + y^2 = 3$, a cylinder of radius $\sqrt{3}$

10. (a) $\mathbf{a} = -\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$

(b) $\mathbf{b} = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$

(c) $\mathbf{a} \times \mathbf{b} = -4\mathbf{k}$

(d) $\mathbf{a} + \mathbf{b} = 2\sqrt{2}\,\mathbf{j}$

(e) $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} \times \mathbf{b}) = 8\sqrt{2}\,\mathbf{i}$

11. (a) $x - y + z = 0$

(b) $\mathbf{u}_1 = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$, $\mathbf{u}_2 = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$

(c) $\mathbf{v} \parallel \mathbf{u}_1 \times \mathbf{u}_2$

12. (a) The line is L: $\mathbf{r}(t) = \langle 0, -2, 0\rangle + t\langle 1, 3, -1\rangle$ and the normal to the plane is $\langle 1, 1, 1\rangle$. Since $\langle 1, 3, -1\rangle$ $\langle 1, 1, 1\rangle = 3 \neq 0$, these vectors are not perpendicular and thus the line intersects the plane.

(b) When $t = 1$, $\mathbf{r}(1) = \langle 1, 1, -1\rangle$ is in the plane $x + y + z = 1$.

13. (a) $P_1 = (1, -1, 0)$, $P_2 = (1, 0, -1)$, $P_3 = (0, -1, 1)$

(b) If $\mathbf{a} = \overrightarrow{P_1P_2} = \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \overrightarrow{P_1P_3} = -\mathbf{i} + \mathbf{k}$, then the area of the triangle is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{\sqrt{3}}{2}$

14. (a) Cylindrical coordinates: $\mathbf{s}(t) = \langle 2, t, 3\rangle$, $0 \leq t \leq 2\pi$

(b) Spherical coordinates: $\mathbf{w}(t) = \left\langle \sqrt{13}, t, \arccos\frac{3}{\sqrt{13}} \right\rangle$, $0 \leq t \leq 2\pi$