

Name:

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- Start by printing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit. Justify your answers.
- No notes, books, calculators, computers or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
Total:		130

Which section-specific problem do you choose? Check exactly one problem. Only this problem can be graded. If you don't commit yourself here, the first attempted problem (of 12a-12e) will be graded.

12a	
12b	
12c	
12d	
12e	

Problem 1) TF questions (20 points) Circle the correct letter. No justifications are needed. All functions or vector fields are assumed to be smooth, which means that arbitrary many partial derivatives exist.

T F

The distance from $(1, 2, -1)$ to $(3, -2, 1)$ is $(-2, 4, -2)$.

T F

The plane $y = 3$ is perpendicular to the xz plane.

T F

All functions $u(x, y)$ that obey $u_x = u$ at all points obey $u_y = 0$ at all points.

T F

The best linear approximation at $(1, 1, 1)$ to the function $f(x, y, z) = x^3 + y^3 + z^3$ is the function $L(x, y, z) = 3x^2 + 3y^2 + 3z^2$

T F

If $f(x, y)$ is any function of two variables, then $\int_0^1 \left(\int_x^1 f(x, y) dy \right) dx = \int_0^1 \left(\int_y^1 f(x, y) dx \right) dy$.

T F

Let $C = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle in the plane and $\mathbf{F}(x, y)$ a vector field satisfying $|\mathbf{F}| \leq 1$. Then $-2\pi \leq \int_C \mathbf{F} \cdot d\mathbf{r} \leq 2\pi$.

T F

A vector field $\mathbf{F} = \langle P(x, y), Q(x, y) \rangle$ is conservative in the plane if and only if $P_y(x, y) = Q_x(x, y)$ for all points (x, y) .

T F

Let \mathbf{a} and \mathbf{b} be two nonzero vectors. Then the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ always point in different directions.

T F

If all the second-order partial derivatives of $f(x, y)$ vanish at (x_0, y_0) then (x_0, y_0) is a critical point of f .

T F

If \mathbf{a}, \mathbf{b} are vectors, then $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram determined by \mathbf{a} and \mathbf{b} .

T F

The distance between two points A, B in space is the length of the curve $\mathbf{r}(t) = A + t(B - A)$, $t \in [0, 1]$.

T F

The function $f(x, y) = xy$ has no critical point.

T F

The length of a curve does not depend on the chosen parameterization.

T F

The equation $\rho = 1$ in spherical coordinates defines a cylinder.

T F

For any numbers a, b satisfying $|a| \neq |b|$, the vector $\langle a - b, a + b \rangle$ is perpendicular to $\langle a + b, b - a \rangle$.

T F

The line integral of $\mathbf{F}(x, y) = \langle -y, x \rangle$ along the counterclockwise oriented boundary of a region R is twice the area of R .

T F

A surface in space for which all normal vectors are parallel to each other must be part of a plane.

T F

There is no surface for which both the parabola and the hyperbola appear as traces.

T F

If $(u, v) \mapsto \mathbf{r}(u, v)$ is a parameterization for a surface, then $\mathbf{r}_u(u, v) + \mathbf{r}_v(u, v)$ is a vector which lies in the tangent plane to the surface.

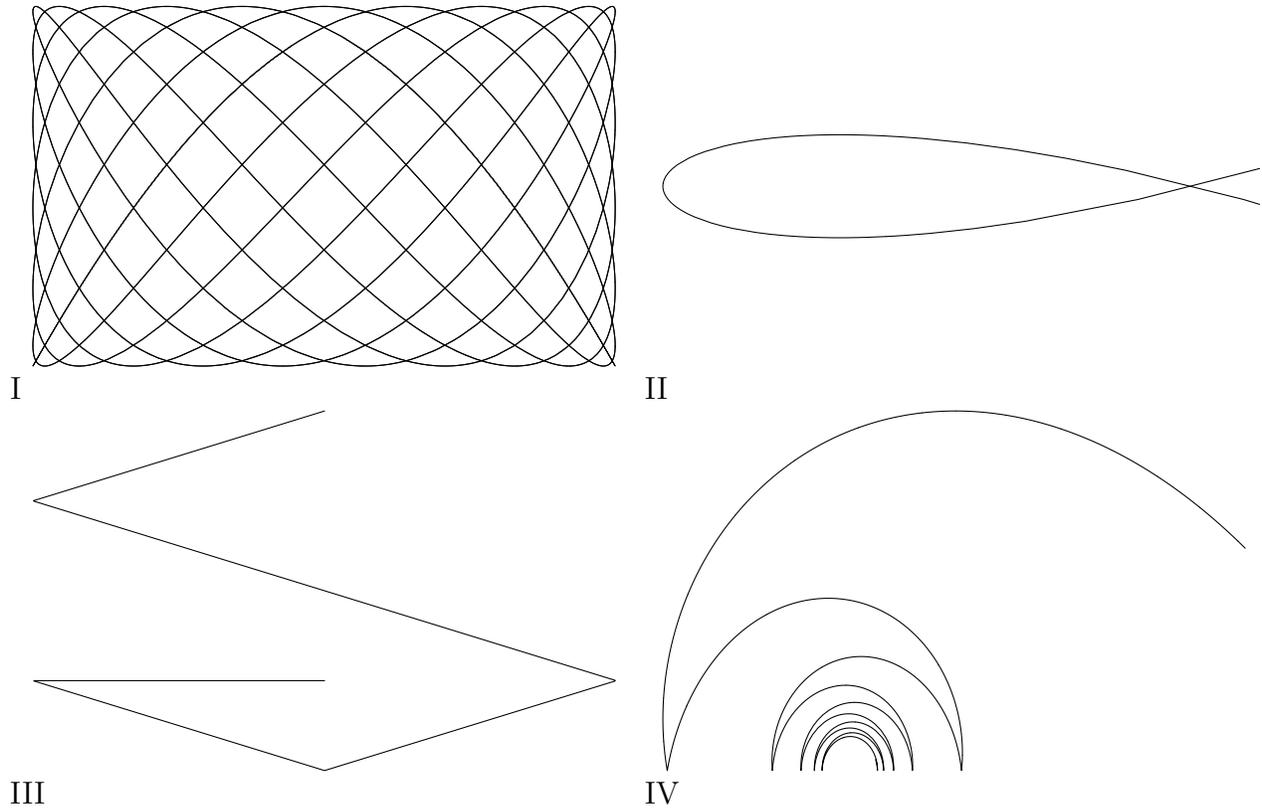
T F

When using spherical coordinates in a triple integral, one needs to include the volume element $dV = \rho^2 \cos(\phi) d\rho d\phi d\theta$.

x 4 =

Problem 2) (10 points)

Match the equations with the curves. No justifications are needed.



Enter I,II,III,IV here	Equation
	$\mathbf{r}(t) = \langle t^2, t^3 - t \rangle$
	$\mathbf{r}(t) = \langle 1 - t , t - t - 1 \rangle$
	$\mathbf{r}(t) = \langle 2 \sin(13t), \cos(22t) \rangle$
	$\mathbf{r}(t) = \langle t \sin(1/t), t \cos(1/t) \rangle$

Problem 3) (10 points)

a) Find an equation for the plane Σ passing through the points $P = (1, 0, 1)$, $Q = (2, 1, 3)$ and $R = (0, 1, 5)$.

b) Find the distance from the origin $(0, 0, 0)$ to Σ .

Problem 4) (10 points)

The equation $f(x, y, z) = e^{xyz} + z = 1 + e$ implicitly defines z as a function $z = g(x, y)$ of x and y .

- Find formulas (in terms of x, y and z) for $g_x(x, y)$ and $g_y(x, y)$.
- Estimate $g(1.01, 0.99)$ using linear approximation.

Problem 5) (10 points)

Find the surface area of the surface S parametrized by $\mathbf{r}(u, v) = \langle u, v, 2 + \frac{u^2}{2} + \frac{v^2}{2} \rangle$ for (u, v) in the disc $\{u^2 + v^2 \leq 1\}$.

Problem 6) (10 points)

Find the local and global extrema of the function $f(x, y) = x^3/3 + y^3/3 - x^2/2 - y^2/2 + 1$ on the disc $D = \{x^2 + y^2 \leq 4\}$.

- Classify every critical point inside the disc $x^2 + y^2 < 4$.
- Find the extrema on the boundary $\{x^2 + y^2 = 4\}$ using the method of Lagrange multipliers.
- Determine the global maxima and minima on all of D .

Problem 7) (10 points)

- Given two nonzero vectors $\mathbf{u} = \langle a, b, c \rangle$ and $\mathbf{v} = \langle d, e, f \rangle$ in \mathbf{R}^3 , write down a formula for the cosine of the angle between them. Find a nonzero vector \mathbf{v} that is perpendicular to $\mathbf{u} = \langle 3, 2, 1 \rangle$. Describe geometrically the set of all \mathbf{v} , including zero, that are perpendicular to this vector \mathbf{u} .
- Consider a function f of three variables. Explain with a picture and a sentence what it means geometrically that $\nabla f(P)$ is perpendicular to the level set of f through P .
- Assume the gradient of f at P is nonzero. Write a few sentences that would convince a skeptic that $\nabla f(P)$ is perpendicular to the level set of f at the point P .
- Assume the level set of f is the graph of a function $g(x, y)$. Explain the relation between the gradient of g and the gradient of f . Especially, how do you relate the orthogonality of ∇f to the level set of f with the orthogonality of ∇g to the level set of g ?

Problem 8) (10 points)

Let R be the region inside the circle $x^2 + y^2 = 4$ and above the line $y = \sqrt{3}$. Evaluate

$$\int \int_R \frac{y}{x^2 + y^2} dA.$$

Problem 9) (10 points)

A region W in \mathbf{R}^3 is given by the relations

$$\begin{aligned}x^2 + y^2 &\leq z^2 \leq 3(x^2 + y^2) \\ 1 &\leq x^2 + y^2 + z^2 \leq 4 \\ x &\geq 0\end{aligned}$$

1. Sketch the region W .
2. Find the volume of the region W .

Problem 10) (10 points)

Consider the vector field

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

defined everywhere in the plane \mathbf{R}^2 except at the origin.

- a) Let C be any closed curve which bounds a region D . Assume that $(0, 0)$ is not contained in D and does not lie on C . Explain why

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

- b) Let C be the unit circle oriented counterclockwise. What is $\int_C \mathbf{F} \cdot d\mathbf{r}$? Explain why your answer shows that there is no function f for which $\mathbf{F}(x, y) = \nabla f(x, y)$ everywhere except at the origin $(0, 0)$.

Problem 11) (10 points)

Let $\mathbf{F}(x, y)$ be a vector field in the plane given by the formula

$$\mathbf{F}(x, y) = \left\langle x^2 - 2xye^{-x^2} + 2y, e^{-x^2} + \frac{1}{\sqrt{y^4 + 1}} \right\rangle.$$

If C is the path which goes from from $(-1, 0)$ to $(1, 0)$ along the semicircle $x^2 + y^2 = 1$, $y \geq 0$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

(Hint: use Green's Theorem.)

SECTION SPECIFIC PROBLEMS

Math 21a

Please choose one of the following problems and register your decision on the first page of this exam.

Problem 12a)

Toss an uneven coin. The probability of getting a head is $1/3$ and the probability getting a tail is $2/3$. One tosses this coin 6 times.

- a) What is the probability of getting exactly 2 heads?
- b) What is the expected number of heads one can get in this experiment? Use random variables to explain.

Problem 12b)

Consider the random variables $X(k) = k$ and $Y(k) = k^2$ on the probability space $\Omega = \{0, 1, 2, 3\}$ with uniform distribution. These random variables can be written as vectors $X = \langle 0, 1, 2, 3 \rangle$, $Y = \langle 0, 1, 4, 9 \rangle$.

- a) Find the expectations of X , Y .
- b) Calculate the covariance $\text{Cov}[X, Y]$ of X and Y .
- c) Determine the regression line $y = ax + b$ for these data.

Problem 12c)

Consider three urns. Urn A contains 2 white and 4 red balls; urn B contains 8 white and 4 red balls; urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white, given that exactly 2 white balls were selected.

Problem 12d)

In appropriate units, the charge density $\sigma(x, y, z)$ in a region in space is given by $\sigma = \nabla \cdot \mathbf{E} = \text{div}(\mathbf{E})$, where \mathbf{E} is the electric field. Consider the cube of side lengths 1 given by $0 \leq x, y, z \leq 1$. What is the total charge in this cube if

$$\mathbf{E} = \langle x(1-x) \log(1+xyz), y(1-y) \tan(x^3 + y^3 + z^3), z(1-z)e^{\sqrt{x+y}} \rangle.$$

(The total charge is the integral of the charge density over the cube.)

Problem 12e)

- a) By calculating the integral $\int \int_S \mathbf{F} \, d\mathbf{S}$ directly, find the flux of the vector field $\mathbf{F}(x, y, z) = \langle 0, 0, x + z \rangle$ through the sphere $x^2 + y^2 + z^2 = 9$, where the sphere is oriented with the normal pointing outward.
- b) Find the flux of the vector field $\mathbf{F}(x, y, z) = \langle 0, 0, x + z \rangle$ through the sphere $x^2 + y^2 + z^2 = 9$ using the divergence theorem.
- c) Explain **in words** without invoking any integral theorem, why the flux integral of the vector field $\mathbf{F}(x, y, z) = \langle 0, 0, x + z \rangle$ through any sphere with positive radius centered at $(0, 0, 0)$ is positive. A one or two sentence explanation is sufficient, but it should be formulated so that it makes sense to somebody who does not know calculus.

Hint: Split up \mathbf{F} as a sum $\mathbf{F} = \langle 0, 0, x \rangle + \langle 0, 0, z \rangle$ and look at the two fluxes separately.