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- Start by printing your name in the above box and check your section in the box to the left.

- Try to answer each question on the same page as the question is asked. If needed, use the back or next empty page for work. If you need additional paper, write your name on it.

- Do not detach pages from this exam packet or unstaple the packet.

- Please write neatly. Answers which are illegible for the grader can not be given credit. Justify your answers.

- No notes, books, calculators, computers or other electronic aids are allowed.

- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
Total:		130

Which section-specific problem do you choose? Check exactly one problem. Only this problem can be graded. If you don't commit yourself here, the first attempted problem (of 12a-12e) will be graded.

12a	
12b	
12c	
12d	
12e	

Problem 1) TF questions (20 points) Circle the correct letter. No justifications are needed.

| T | F | The distance from $(1, 2, -1)$ to $(3, -2, 1)$ is $(-2, 4, -2)$. |

False. The distance is a number, not a vector.

| T | F | The plane $y = 3$ is perpendicular to the xz plane. |

False. The plane is parallel to the xz plane.

| T | F | All functions $u(x, y)$ that obey $u_x = u$ at all points obey $u_y = 0$ at all points. |

False. The function $u(x, y) = e^x y$ satisfies $u_x = u$ but $u_y = e^x$.

| T | F | The best linear approximation at $(1, 1, 1)$ to the function $f(x, y, z) = x^3 + y^3 + z^3$ is the function $L(x, y, z) = 3x^2 + 3y^2 + 3z^2$ |

False. The linear approximation is a linear function in x, y, z. The correct linear approximation would here be $L(x, y, z) = 0$.

| T | F | If $f(x, y)$ is any function of two variables, then $\int_0^1 \left(\int_x^1 f(x, y) \, dy \right) dx = \int_0^1 \left(\int_y^1 f(x, y) \, dx \right) dy$. |

False. The correct identity would be $\int_0^1 \left(\int_x^1 f(x, y) \, dy \right) dx = \int_0^1 \left(\int_0^y f(x, y) \, dx \right) dy$.

| T | F | Let $C = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1 \}$ be the unit circle in the plane and $\mathbf{F}(x, y)$ a vector field satisfying $|\mathbf{F}| \leq 1$. Then $-2\pi \leq \int_C \mathbf{F} \cdot dr \leq 2\pi$. |

True. By definition $\int \mathbf{F} \cdot dr = \int_0^{2\pi} F(r(t)) \cdot r'(t) \, dt$ and so $| \int \mathbf{F} \cdot dr| = \int_0^{2\pi} |F(r(t)) \cdot r'(t)| \, dt \leq \int_0^{2\pi} |\mathbf{F}(r(t))||r'(t)| \, dt \leq \int_0^{2\pi} |r'(t)| \, dt = 2\pi$.

| T | F | A vector field $\mathbf{F} = \langle P(x, y), Q(x, y) \rangle$ is conservative in the plane if and only if $P_y(x, y) = Q_x(x, y)$ for all points (x, y). |

True. This is a consequence of Green's theorem: the line integral along a closed curve is the double integral of $Q_x - P_y$ over the enclosed region and so zero. The reverse is easier to see: if a potential f satisfying $\nabla f = (f_x, f_y) = (P, Q)$ exists, then $Q_x - P_y = f_{yx} - f_{xy} = 0$ (Clairot).

| T | F | Let $\mathbf{a}$ and $\mathbf{b}$ be two nonzero vectors. Then the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ always point in different directions. |

False. Take $\mathbf{a} = \langle 4, 2 \rangle$ and $\mathbf{b} = \langle 2, 1 \rangle$. Then $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ point in the same direction.

| T | F | If all the second-order partial derivatives of $f(x, y)$ vanish at (x_0, y_0) then (x_0, y_0) is a critical point of f. |

False. Take $f(x, y) = x + x^3 + y^3$. Then $(0, 0)$ not a critical point even so all second-order partial derivatives are zero.

| T | F | If $\mathbf{a}, \mathbf{b}$ are vectors, then $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$. |

True. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|| \sin(\phi)|$. $|\mathbf{a}|$ is the length of the base of the parallelogram and $|\mathbf{b}|| \sin(\phi)|$ is the height of the parallelogram.

T	F

The distance between two points A, B in space is the length of the curve $\mathbf{r}(t) = A + t(B - A)$, $t \in [0, 1]$.

True. $|\mathbf{r}'(t)| = |B - A|$ and $\int_0^1 |r'(t)| \, dt = |B - A|$.

T	F

The function $f(x, y) = xy$ has no critical point.

False. $(0, 0)$ is a critical point of f.

T	F

The length of a curve does not depend on the chosen parameterization.

True. This is a consequence of the chain rule: if $r(s(t))$ is a new parameterization, then $\int_a^b |r(s(t))'| \, dt = \int_a^b |r'(s(t))||s'(t)|dt = \int_{s(a)}^{s(b)} |r'(s)|ds$.

T	F

The equation $\rho = 1$ in spherical coordinates defines a cylinder.

False. It defines a sphere.

T	F

For any numbers a, b satisfying $|a| \neq |b|$, the vector $\langle a - b, a + b \rangle$ is perpendicular to $\langle a + b, b - a \rangle$.

True. The dot product is 0.

T	F

The line integral of $\mathbf{F}(x, y) = \langle -y, x \rangle$ along the counterclockwise oriented boundary of a region R is twice the area of R.

True. The curl of F is constant 2. By Greens theorem, the line integral is $\int \int_R 2 \, dA$, twice the area of R.

T	F

A surface in space for which all normal vectors are parallel to each other must be part of a plane.

True. Fix a point $\mathbf{x}_0$ and let $\mathbf{n} = (a, b, c)$ be the vector to which all normal vectors are parallel, then by assumption for any point $\mathbf{x}$, one has by assumption $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n} = 0$ which is the equation of a plane.

T	F

There is no surface for which both the parabola and the hyperbola appear as traces.

False. The hyperbolic paraboloid is an example.

T	F

If $(u, v) \mapsto \mathbf{r}(u, v)$ is a parameterization for a surface, then $\mathbf{r}_u(u, v) + \mathbf{r}_v(u, v)$ is a vector which lies in the tangent plane to the surface.
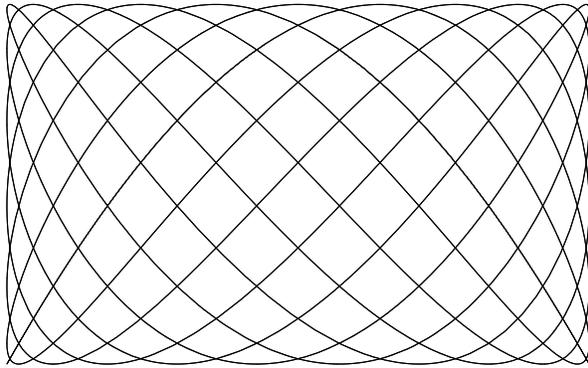
True Both $\mathbf{r}_u(u, v)$ as well as as $\mathbf{r}_v(u, v)$ are tangent to the surface. Therefore, also the sum is tangent.

T	F

When using spherical coordinates in a triple integral, one needs to include the volume element $dV = \rho^2 \cos(\phi) \, d\rho d\phi d\theta$.
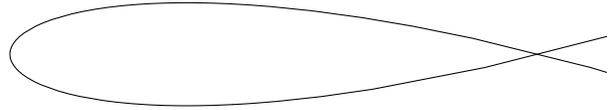
False The correct factor would be $\rho^2 \sin(\phi)$.
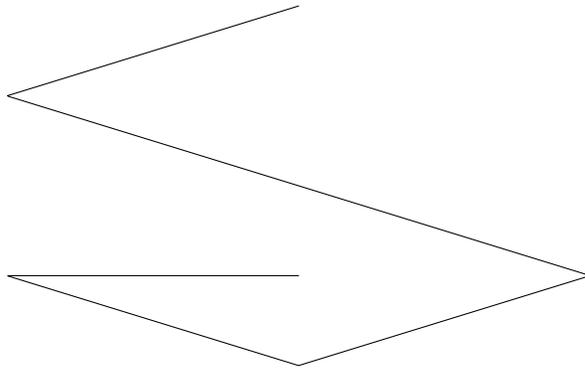
Problem 2) (10 points)

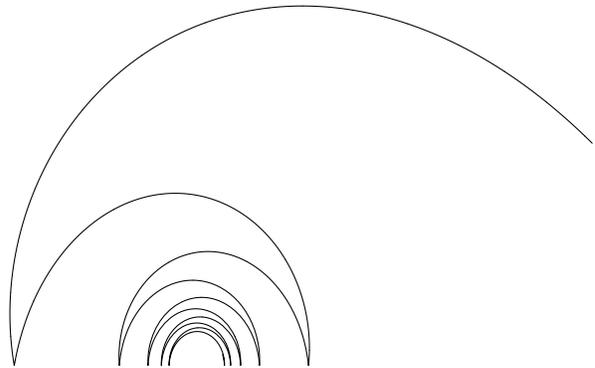Match the equations with the curves. No justifications are needed.

I

II

III

IV

Enter I,II,III,IV here	Equation
II	$\mathbf{r}(t) = \langle t^2, t^3 - t \rangle$
III	$\mathbf{r}(t) = \langle |1 - |t||, |t - |t - 1|| \rangle$
I	$\mathbf{r}(t) = \langle 2\sin(13t), \cos(22t) \rangle$
IV	$\mathbf{r}(t) = \langle t\sin(1/t), t|\cos(1/t)| \rangle$

Problem 3) (10 points)

a) Find an equation for the plane Σ passing through the points $P = (1, 0, 1)$, $Q = (2, 1, 3)$ and $R = (0, 1, 5)$.

b) Find the distance from the origin $(0, 0, 0)$ to Σ.

Solution.

a) Take the cross product between two vectors in the plane to get a normal vector $n = (a, b, c) = (1, 1, 2) \times (-1, 1, 4) = (2, -6, 2)$. The plane has the equation $ax + by + cz = d$, where d is obtained from plugging in one of the points. The answer is $\boxed{x - 3y + z = 2.}$

b) The distance is $|(1, 0, 1) \cdot \mathbf{n}|/|n| = \boxed{2/\sqrt{11}}$

Problem 4) (10 points)

The equation $f(x, y, z) = e^{xyz} + z = 1 + e$ implicitly defines z as a function $z = g(x, y)$ of x and y.

a) Find formulas (in terms of x, y and z) for $g_x(x, y)$ and $g_y(x, y)$.

b) Estimate $g(1.01, 0.99)$ using linear approximation.

Solution.

a) By the chain rule $g_x = -f_x/f_z = -yze^{xyz}/(1 + xye^{xyz})$ and $g_y = -f_y/f_z = -xze^{xyz}/(1 + xye^{xyz})$.

b) Note that $f(1, 1, 1) = 1 + e$ so that $(1, 1, 1)$ is on the surface and $g(1, 1) = 1$. From a) we know $g_x(1, 1) = -e/(1 + e)$ and $g_y(1, 1) = -e/(1 + e)$. By linearization, $g(1 + a, 1 + b) = g(1, 1) + g_x(1, 1)a + g_y(1, 1)b = 1 - (a + b)e/(1 + e)$. In our case, where $a = 0.01, b = -0.01$ we estimate $g(1.01, 0.99) = 1$.

Problem 5) (10 points)

Find the surface area of the surface S parametrized by $\mathbf{r}(u, v) = \langle u, v, 2 + \frac{u^2}{2} + \frac{v^2}{2} \rangle$ for (u, v) in the disc $D = \{u^2 + v^2 \leq 1\}$.

Solution.

$r_u = (1, 0, u), r_v = (0, 1, v)$ and $r_u \times r_v = (-u, -v, 1)$.

The surface area is $\int \int_D \sqrt{1 + u^2 + v^2} \; du dv = \int_0^1 \int_0^{2\pi} \sqrt{1 + r^2} r \; d\theta dr = 2\pi \int_0^1 \sqrt{1 + r^2} r \; dr = 2\pi(1 + r^2)^{3/2}/3|_0^1 = \boxed{2\pi(\sqrt{8} - 1)/3}$.

Problem 6) (10 points)

Find the local and global extrema of the function $f(x, y) = x^3/3 + y^3/3 - x^2/2 - y^2/2 + 1$ on the disc $\{x^2 + y^2 \leq 4\}$.

a) Classify every critical point inside the disc $x^2 + y^2 < 4$.

b) Find the extrema on the boundary $\{x^2 + y^2 = 4\}$ using the method of Lagrange multipliers.

c) Determine the global maxima and minima on all of D.

Solution.
a) $\nabla f = (x^2 - x, y^2 - y)$. The critical points of f are $(0,0), (0,1), (1,0), (1,1)$.
The Hessian determinant (=discriminant) is $D = (2x - 1)(2y - 1)$, which is $1, -1, -1, 1$. The point $(0,0)$ is a local maximum, the point $(1,1)$ is a local minimum and $(0,1), (1,0)$ are saddle points.

b) $g(x,y) = x^2 + y^2 - 4$. The Lagrange equations are

$x^2 - x = \lambda 2x$
$y^2 - y = \lambda 2y$
$x^2 + y^2 - 4 = 0$

which have solutions
$\boxed{(-2,0), (0,-2), (0,2), (2,0), (-\sqrt{2}, -\sqrt{2}), (\sqrt{2}, \sqrt{2})}$. At these points, the function f takes values $-14/3 + 1, -14/3 + 1, 2/3 + 1, 2/3 + 1, -\sqrt{2} + 1, -2 - 4\sqrt{2}/3 + 1, -2 + 4\sqrt{2}/3 + 1$.
c) To find the maximum and minimum, just compare the values at all the candidates obtained in a) and b):
$\boxed{\text{the minimum is at the two points } (-2,0), (0,-2)}$
where f takes the value $-42/3 + 1$.
$\boxed{\text{The maximum is at the points } (0,2), (2,0)}$
where f takes the value $2/3 + 1$.

$\boxed{\text{Problem 7) (10 points)}}$
a) Given two nonzero vectors $\mathbf{u} = \langle a, b, c \rangle$ and $\mathbf{v} = \langle d, e, f \rangle$ in $\mathbf{R}^3$, write down a formula for the cosine of the angle between them. Find a nonzero vector $\mathbf{v}$ that is perpendicular to $\mathbf{u} = \langle 3, 2, 1 \rangle$. Describe geometrically the set of all $\mathbf{v}$, including zero, that are perpendicular to this vector $\mathbf{u}$.

b) Consider a function f of three variables. Explain with a picture and a sentence what it means geometrically that $\nabla f(P)$ is perpendicular to the level set of f through P.

c) Assume the gradient of f at P is nonzero. Write a few sentences that would convince a skeptic that $\nabla f(P)$ is perpendicular to the level set of f at the point P.

d) Assume the level set of f is the graph of a function $g(x,y)$. Explain the relation between the gradient of g and the gradient of f. Especially, how do you relate the orthogonality of ∇f to the level set of f with the orthogonality of ∇g to the level set of g?

Solution.
a) $\cos(\alpha) = ad + be + cf/(\sqrt{a^2 + b^2 + c^2}\sqrt{d^2 + e^3 + f^2})$. The vector $(1, -1, -1)$ is perpendicular to $(3, 2, 1)$. The set of points which are perpendicular to $(3, 2, 1)$ satisfies the equation $3x + 2y + z = 0$.

b) ∇f is orthogonal to every tangent vector to the level surface. ∇f is orthogonal to the velocity vector of a curve $r(t)$ on the level surface.

c) Take two curves on the level surface which cross transversally. Because f is constant on the level surface, $f(r(t)) = c$ is constant or by the chain rule, $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t) = 0$. This means that ∇f is orthogonal to the velocity vector $r'(t)$. By b), ∇f is orthogonal to the level surface.

d) In the special case $f(x, y, z) = g(x, y) - z$, we have $\nabla f = (\nabla g, -1)$. We see that projecting the vector ∇f onto the xy plane gives the vector ∇g.

Problem 8) (10 points)

Let R be the region inside the circle $x^2 + y^2 = 4$ and above the line $y = \sqrt{3}$. Evaluate

$$\int \int_R \frac{y}{x^2 + y^2} \, dA .$$

Solution. The key is to set up the integral in polar coordinates:

$$\int_{\pi/3}^{2\pi/3} \int_{\sqrt{3}/\sin(\theta)}^{2} \frac{r \sin(\theta)}{r^2} \, r \, dr \, d\theta .$$

This gives $\int_{\pi/3}^{2\pi/3} (2\sin(\theta) - \sqrt{3}) \, d\theta = (-2\cos(\theta) - \sqrt{3}\theta)|_{\pi/3}^{2\pi/3} = -2(-1/2 - 1/2) - \sqrt{3}\pi/3 = \boxed{2 - \sqrt{3}\pi/3}$.

Problem 9) (10 points)

A region W in $\mathbf{R}^3$ is given by the relations

$$x^2 + y^2 \leq z^2 \leq 3(x^2 + y^2)$$
$$1 \leq x^2 + y^2 + z^2 \leq 4$$
$$x \geq 0$$

1. Sketch the region W.

2. Find the volume of the region W.

Solution.

1. First inequalities: the region is sandwiched between two twosided cones. Second inequalities: the region is sandwiched between two spheres.

2. Use spherical coordinates. The restriction $x \geq 0$ gives half of the region. Because also the restriction $z \geq 0$ gets half of the region, we can get the volume

$$\int_1^2 \int_{\pi/6}^{\pi/4} \int_0^{2\pi} \rho^2 \sin(\phi) \, d\theta \, d\phi \, d\rho$$

which is $\boxed{(7/3)\pi(\sqrt{3} - \sqrt{2})}$.

Consider the vector field

$$\mathbf{F}(x, y) = \langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \rangle$$

defined everywhere in the plane $\mathbf{R}^2$ except at the origin.

a) Let C be any closed curve which bounds a region D. Assume that $(0,0)$ is not contained in D and does not lie on C. Explain why

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0 .$$

b) Let C be the unit circle oriented counterclockwise. What is $\int_C \mathbf{F} \cdot d\mathbf{r}$? Explain why your answer shows that there is no function f for which $\mathbf{F}(x, y) = \nabla f(x, y)$ everywhere except at the origin $(0,0)$.

Solution.
a) F is conservative in the region $\mathbf{R}^2 \setminus \{(0,0)\}$ because with $F = (P, Q)$ we have $Q_x - P_y = 0$ and Greens theorem assures that the line integral along any closed curve is zero, provided the region does not contain 0.

b) A direct calculation with $r(t) = (x(t), y(t)) = (\cos(t), \sin(t))$ gives $\int_0^{2\pi} 1 \ dt = 2\pi$. If there would exist a potential function f defined all over the plane, then the line integral along any closed curve would be zero. This would contradict the result for the curve along the unit circle.

Let $\mathbf{F}(x, y)$ be a vector field in the plane given by the formula

$$\mathbf{F}(x, y) = \langle x^2 - 2xye^{-x^2} + 2y, e^{-x^2} + \frac{1}{\sqrt{y^4 + 1}} \rangle .$$

If C is the path which goes from from $(-1, 0)$ to $(1, 0)$ along the semi circle $x^2 + y^2 = 1$, $y \geq 0$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.
(Hint: use Green's Theorem.)

Solution. The curl of F is -2. The line integral along any curve from $(-1, 0)$ to $(1, 0)$ minus the line integral take the curve $r(t) = (t, 0)$ for $t \in [-1, 1]$ is by Green's theorem $\int \int_D 1 \ dxdy = \pi$. The line integral along the line segment is $\int_{-1}^1 t^2 \ dt = \boxed{2/3}$. Therefore, the result is $\boxed{2/3 - \pi}$.

SECTION SPECIFIC PROBLEMS   Math 21a

Please choose one of the following problems and register your decision on the first page of this exam.

Toss an uneven coin. The probability of getting a head is 1/3 and the probability getting a tail is 2/3. One tosses this coin 6 times.

a) What is the probability of getting exactly 2 heads?

b) What is the expected number of heads one can get in this experiment? Use random variables to explain.

Solution.
a) Use the Bernoulli formula: $(1/3)^2(2/3)^4 \frac{6!}{(2!4!)} = \boxed{80/243}$.

b) $X[(\omega_1, ..\omega_6)] = \omega_1 + ... + \omega_6 = \sum_{i=1}^{6} X_i$. Because the random variables X_i are independent, and the expected number of heads with one coin is in each case $E[X_i] = 1 \cdot 1/3 + 0 \cdot 2/3 = 1/3$, the expectation number of heads is $6/3 = \boxed{2}$.

Problem 12b)

Consider the random variables $X(i) = i$ and $Y(i) = i^2$ on the probability space $\Omega = \{0, 1, 2, 3\}$ with uniform distribution. These random variables can be written as vectors $X = \langle 0, 1, 2, 3 \rangle, Y = \langle 0, 1, 4, 9 \rangle$.

a) Find the expectations of X, Y.

b) Calculate the covariance $\text{Cov}[X, Y]$ of X and Y.

c) Determine the regression line $y = ax + b$ for these data.

Solution.
a) $E[X] = (0 + 1 + 2 + 3)/4 = \boxed{3/2}$. $E[Y] = (0 + 1 + 4 + 9)/4 = \boxed{7/2}$.

b) $\text{Cov}[X, Y] = E[X - E[X], Y - Y] = \boxed{15/4}$.

c) $y = ax + b = 3x - 1$. $\boxed{y = 3x - 1}$.

Problem 12c)

Consider three urns. Urn A contains 2 white and 4 red balls; urn B contains 8 white and 4 red balls; urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white, given that exactly 2 white balls were selected.

Solution. Look at the events U that the ball from urn A is white and V that two white balls were chosen. We have to compute $P[U|V] = P[U \cap V]/P[V]$. There are 3 possibilities that two white balls are chosen:
1) $[A(w), B(w), C(r)]$ with probability $(2/6)(8/12)(3/4) = 1/6$.
2) $[A(w), B(r), C(w)]$ with probability $(2/6)(4/12)(1/4) = 1/36$.
3) $[A(r), B(w), C(w)]$ with probability $(4/6)(8/12)(1/4) = 1/9$.

The probability of the event V is $11/36$. The probability of $U \cap V$ is $1/6 + 1/36 = 7/36$. The answer is $(7/36)/(11/36) = \boxed{7/11}$.

Problem 12d)

In appropriate units, the charge density $\sigma(x, y, z)$ in a region in space is given by $\sigma = \nabla \cdot \mathbf{E} = \text{div}(\mathbf{E})$, where $\mathbf{E}$ is the electric field. Consider the cube of side lengths 1 given by $0 \leq x, y, z \leq 1$. What is the total charge in this cube if

$$\mathbf{E} = \langle x(1-x)\log(1+xyz), y(1-y)\tan(x^3 + y^3 + z^3), z(1-z)e^{\sqrt{x+y}} \rangle .$$

(The total charge is the integral of the charge density over the cube.)

Solution. On the x-faces, we know that $F = (0, Q, R)$. The flux through the x-faces is 0. Similarly, the fluxes through the other sides is zero. By the divergence theorem, the triple integral on the unit cube is $\boxed{0}$.

Problem 12e)

a) By calculating the integral $\int\int_S \mathbf{F} \, d\mathbf{S}$ directly, find the flux of the vector field $\mathbf{F}(x, y, z) = \langle 0, 0, x + z \rangle$ through the sphere $x^2 + y^2 + z^2 = 9$, where the sphere is oriented with the normal pointing outward.

b) Find the flux of the vector field $\mathbf{F}(x, y, z) = \langle 0, 0, x + z \rangle$ through the sphere $x^2 + y^2 + z^2 = 9$ using the divergence theorem.

c) Explain **in words** without invoking any integral theorem, why the flux integral of the vector field $\mathbf{F}(x, y, z) = \langle 0, 0, x + z \rangle$ through any sphere with positive radius centered at $(0, 0, 0)$ is positive. A one or two sentence explanation is sufficient, but it should be formulated so that it makes sense to somebody who does not know calculus.

Hint: Split up $\mathbf{F}$ as a sum $\mathbf{F} = \langle 0, 0, x \rangle + \langle 0, 0, z \rangle$ and look at the two fluxes separately.

Solution.
a) $\int_0^{2\pi} \int_0^{\pi} 9 \cos(\phi) \sin(\phi) \, d\phi d\theta = \boxed{36\pi}$.

b) The divergence of $\mathbf{F}$ is 1. The integral of the divergence over the sphere of radius R is $4\pi R^3/3 = \boxed{36\pi}$.

c) The flux integral of $F(x, y, z) = (0, 0, x)$ is zero by symmetry: there is the same flux on upper and lower hemisphere with opposite sign. The flux integral of $F(x, y, z) = (0, 0, z)$ is larger through the upper then through the lower hemisphere.