SOLUTIONS TO PRACTICE EXAM FOR FINAL: 1/13/2002 **Math 21a**

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- Start by printing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit. Justify your answers.
- No notes, books, calculators, computers or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
Total:		130

Which section-specific problem do you choose? Check exactly one problem. Only this problem can be graded. If you don't commit yourself here, the first attempted problem will be graded.

12a	
12b	
12c	
12d	

Problem 1) TF questions (20 points) Circle the correct letter. No justifications are needed.

| T | F |

The length of the curve $\mathbf{r}(t) = (\sin(t), t^4 + t, \cos(t))$ on $t \in [0, 1]$ is the same as the length of the curve $\mathbf{r}(t) = (\sin(t^2), t^8 + t^2, \cos(t^2))$ on $[0, 1]$.

True. This is a consequence of the chain rule: $\int_a^b |r(s(t))'|\, dt = \int_a^b |r'(s(t))||s'(t)|dt = \int_{s(a)}^{s(b)} |r'(s)|ds$.

| T | F |

The parametric surface $\mathbf{r}(u, v) = (5u - 3v, u - v - 1, 5u - v - 7)$ is a plane.

True. The coordinate functions $r(u, v) = (x(u, v), y(u, v), z(u, v))$ are all linear.

| T | F |

Any function $u(x, y)$ that obeys the differential equation $u_{xx} + u_x - u_y = 1$ has no local maxima.

True. If $\nabla u = (u_x, u_y) = (0, 0)$, then $u_{xx} = 1$ which is incompatible with a local maximum, where $u_{xx} > 0$ by the second derivative test.

| T | F |

The scalar projection of a vector $\mathbf{a}$ onto a vector $\mathbf{b}$ is the length of the vector projection of $\mathbf{a}$ onto $\mathbf{b}$.

True. By definition.

| T | F |

If $f(x, y)$ is a function such that $f_x - f_y = 0$ then f is conservative.

False. The notion of conservative applies to vector fields and not to functions.

| T | F |

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v}$ for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

False. While $|(\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v}|$ and $|(\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v}|$ are both equal to the volume of the parallelepiped determined by $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, the sign is different. An example: for $\mathbf{u} = \langle 1, 0, 0\rangle, \mathbf{v} = \langle 0, 1, 0\rangle, \mathbf{w} = \langle 0, 0, 1\rangle$, we have $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 1$ and $(\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v} = -1$.

| T | F |

The equation $\rho = \phi/4$ in spherical coordinates is half a cone.

False. The equation $\rho = \phi/4$ defines a heart shaped rotational symmetric surface. The surface $\phi = c = const$ would define half a cone for $c \in [0, \pi]$.

| T | F |

The function $f(x, y) = \begin{cases} \frac{x}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ is continuous at every point in the plane.

False. Taking $y = 0$, we get $f(x, 0) = 1/x$ which is discontinuous.

T	F	$\int_0^1 \int_0^x 1 \, dydx = 1/2$.

True. This is the area of half of the unit square.

T	F	Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors which are perpendicular to a given plane Σ. Then $\mathbf{a} + \mathbf{b}$ is also perpendicular to Σ.

True. If $\mathbf{v}$ is a vector in the plane, then $\mathbf{a} \cdot \mathbf{v} = 0$ and $\mathbf{b} \cdot \mathbf{v} = 0$ then also $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{v} = 0$.

T	F	If $g(x, t) = f(x - vt)$ for some function f of one variable $f(z)$ then g satisfies the differential equation $g_{tt} - v^2 g_{xx} = 0$.

True. Actually one could show that $g(x, t) = f(x - vt) + h(x + vt)$ is the general solution of the wave equation $g_{tt} - v^2 g_{xx} = 0$.

T	F	If $f(x, y)$ is a continuous function on $\mathbf{R}^2$ such that $\int \int_D f \, dA \geq 0$ for any region D then $f(x, y) \geq 0$ for all (x, y).

True. Assume $f(a, b) < 0$ at some point (a, b), then $f(x, y) < 0$ in a small neighborhood D of (a, b) and also $\int \int_D f \, dA < 0$ contradicting the assumption.

T	F	Assume the two functions $f(x, y)$ and $g(x, y)$ have both the critical point $(0, 0)$ which are saddle points, then $f + g$ has a saddle point at $(0, 0)$.

False. Example $f(x, y) = x^2 - y^2/2, g(x, y) = -x^2/2 + y^2$ have both a saddle point at $(0, 0)$ but $f + g = x^2/2 + y^2/2$ has a minimum at $(0, 0)$.

T	F	If $f(x, y)$ is a function of two variables and if $h(x, y) = f(g(y), g(x))$, then $h_x(x, y) = f_y(g(y), g(x))g'(y)$.

False. The correct identity would be $h_x(x, y) = f_y(g(y), g(x))g'(x)$ according to the chain rule.

T	F	If we rotate a line around the z axes, we obtain a cylinder.

False. The surface could also be a one-sheeted hyperboloid or a cone.

T	F	The line integral of $\mathbf{F}(x, y) = (x, y)$ along an ellipse $x^2 + 2y^2 = 1$ is zero.

True. The curl $Q_x - P_y$ of the vector field $\mathbf{F}(x, y) = (P, Q)$ is 0. By Green's theorem, the line integral is zero. An other way to see this is that F is a gradient field $F = \nabla f$ with $f(x, y) = (x^2 + y^2)/2$. Therefore F is conservative: the line integral along any closed curve in the plane is zero.

T	F

If $u(x, y)$ satisfies the transport equation $u_x = u_y$, then the vector field $\mathbf{F}(x, y) = \langle u(x, y), u(x, y) \rangle$ is a gradient field.

True. $\mathbf{F} = (P, Q) = (u, u)$. From $u_x = u_y$ we get $Q_x = P_y$ which implies that F is a gradient field.

T	F

$3\ \mathrm{grad}(f) = \frac{d}{dt} f(x + t, y + t, z + t)$.

False. The left hand side is a vector field, the right hand side a function.

T	F

$\int_0^1 \int_0^{2\pi/11} \int_0^{\pi} \rho^2 \sin(\phi)\ d\phi d\theta d\rho = 4\pi/33$.

True. The region is a "lemon slice" which is 1/11'th of a sphere.

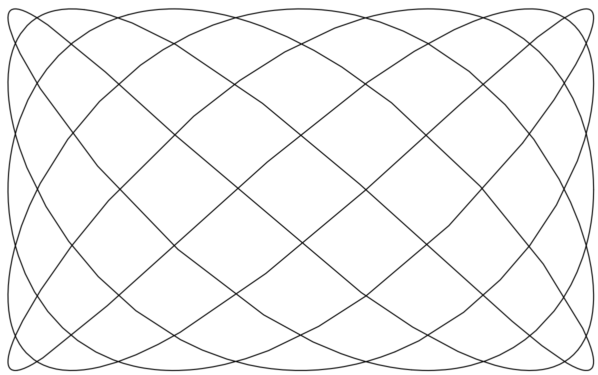
T	F

If $\mathbf{F}$ is a vector field in space and f is equal to the line integral of $\mathbf{F}$ along the straight line C from $(0, 0, 0)$ to (x, y, z), then $\nabla f = \mathbf{F}$.

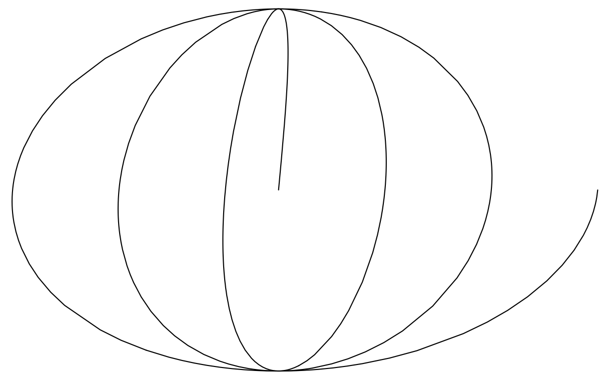
False. This would be true if $\mathbf{F}$ were a conservative vector field. In that case, f would be a potential. In general this is false: for example if $\mathbf{F}(x, y, z) = (0, x, 0)$, then $\int_C \mathbf{F} \cdot \mathbf{dr} = x^2/2$ and $\nabla f(x, y, z) = (x, 0, 0)$ which is different from $\mathbf{F}$.

Problem 2) (10 points)

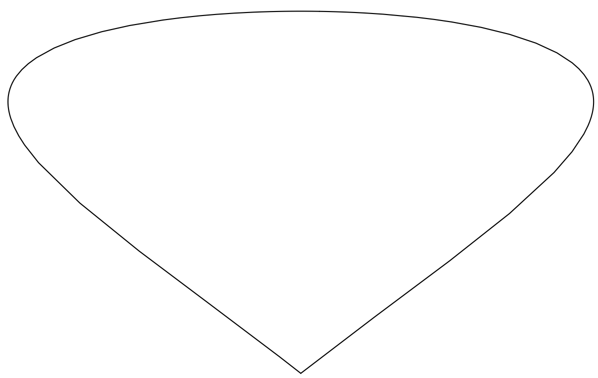
Match the equations with the curves. No justifications are needed.



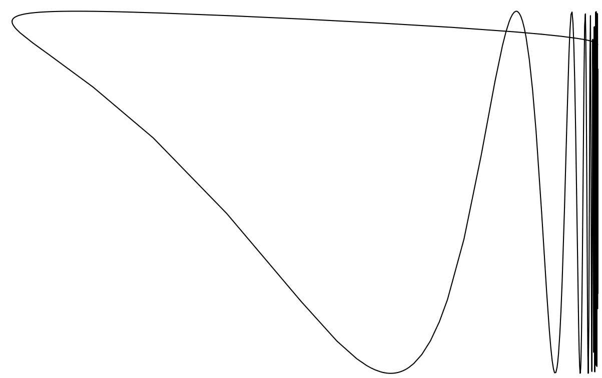
I



II



III



IV

Enter I,II,III,IV here	Equation
III	$\mathbf{r}(t) = (\sin(t), t(2\pi - t))$
I	$\mathbf{r}(t) = (\cos(5t), \sin(7t))$
II	$\mathbf{r}(t) = (t\cos(t), \sin(t))$
IV	$\mathbf{r}(t) = (\cos(t), \sin(6/t))$

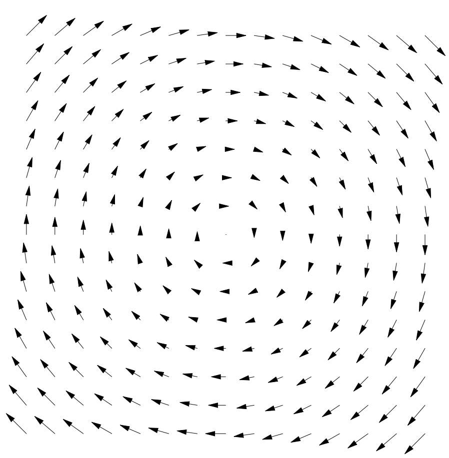
Problem 3) (10 points)

In this problem, vector fields F are written as $F = (P, Q)$. We use abbreviations $\text{curl}(F) = Q_x - P_y$ and $\text{div}(F) = P_x + Q_y$. When stating $\text{curl}(F)(x, y) = 0$ we mean that $\text{curl}(F)(x, y) = 0$ vanishes for **all** (x, y). The statement $\text{curl}(F) \neq 0$ means that $\text{curl}(F)(x, y)$ does not vanish for at least one point (x, y). The same remark applies if curl is replaced by div.

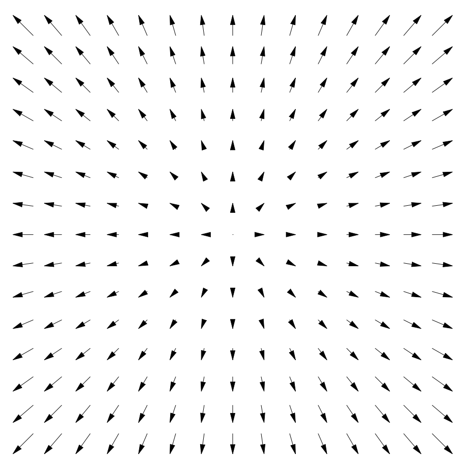
Check the box which match the formulas of the vectorfields with the corresponding picture I,II,III or IV. Mark also the places, indicating the vanishing or not vanishing of curl and div. In each of the four lines, you should finally have circled three boxes. No justifications are needed.

Vectorfield	I	II	III	IV	$\text{curl}(F) = 0$	$\text{curl}(F) \neq 0$	$\text{div}(F) = 0$	$\text{div}(F) \neq 0$
$\mathbf{F}(x, y) = (0, 5)$			X		X		X	
$\mathbf{F}(x, y) = (y, -x)$	X					X	X	
$\mathbf{F}(x, y) = (x, y)$		X			X			X
$\mathbf{F}(x, y) = (2, x)$				X		X	X	

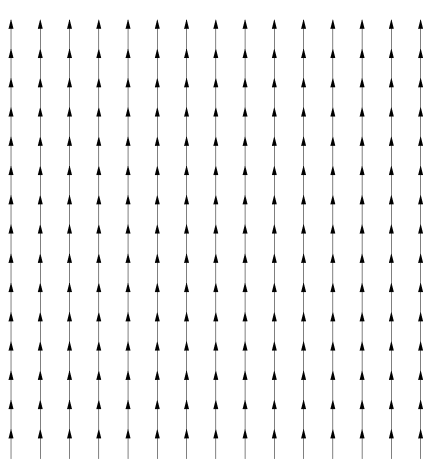
I



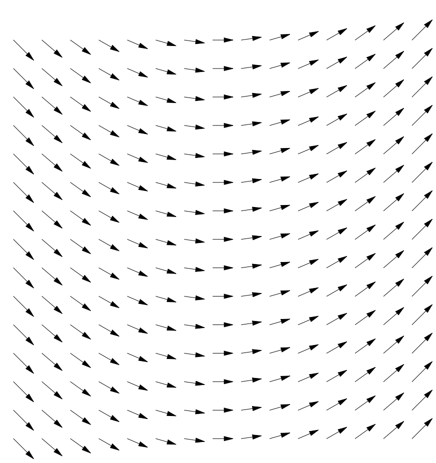
II



III



IV



Problem 4) (10 points)

a) Find the scalar projection of the vector $\mathbf{v} = (3, 4, 5)$ onto the vector $\mathbf{w} = (2, 2, 1)$.

b) Find the equation of a plane which contains the vectors $\langle 1, 1, 0\rangle$ and $\langle 0, 1, 1\rangle$ and contains the point $(0, 1, 0)$.

Solution.
a) $|\mathbf{v} \cdot \mathbf{w}|/|\mathbf{w}| = (6 + 8 + 5)/3 = \boxed{19/3}$
b) $(1, 1, 0) \times (0, 1, 1) = (1, -1, 1)$. The plane has the form $x - y + z = d$ and $d = -1$ is obtained by plugging in the point $(0, 1, 0)$. The solution is $\boxed{x - y + z = -1}$

Problem 5) (10 points)

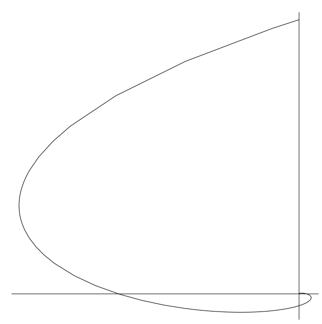
Find the surface area of the ellipse cut from the plane $z = 2x+2y+1$ by the cylinder $x^2+y^2 = 1$.

Solution.
Parameterize the surface $r(u, v) = (u, v, 2u + 2v + 1)$ on the disc $R = \{u^2 + y^2 \leq 1\}$. We get $r_u \times r_v = \langle 1, 0, 2\rangle \times \langle 0, 1, 2\rangle = \langle -2, -2, 1\rangle$ and $|r_u \times r_v| = 3$. The surface integral $\int\int_R |r_u \times r_v| \; dudv = \int\int_R 3 \; dudv = 3\int\int_R \; dudv$ which is 3 times the area of the disc R: Solution: $\boxed{3\pi}$.

Problem 6) (10 points)

Sketch the plane curve $\mathbf{r}(t) = (\sin(t)e^t, \cos(t)e^t)$ for $t \in [0, 2\pi]$ and find its length.

Solution.
$\mathbf{r}'(t) = \langle \cos(t)e^t + \sin(t)e^t, -\sin(t)e^t + \cos(t)e^t\rangle$ satisfies $|\mathbf{r}'(t)| = \sqrt{2}e^t$ so that $\int_0^{2\pi} |\mathbf{r}'(t)| \; dt = \boxed{\sqrt{2}(e^{2\pi} - 1)}$.



Problem 7) (10 points)

Let $f(x, y, z) = 2x^2+3xy+2y^2+z^2$ and let R denote the region in $\mathbf{R}^3$, where $2x^2+2y^2+z^2 \leq 1$. Find the maximum and minimum values of f on the region R and list all points, where said maximum and minimum values are achieved. Distinguish between local extrema in the interior and extrema on the boundary.

a) Extrema in the interior of the ellipsoid $2x^2 + 2y^2 + z^2 < 1$.
$\nabla f(x, y, z) = \langle 4x + 3y, 3x + 4y, 2z\rangle = \langle 0, 0, 0\rangle$ for $(x, y, z) = (0, 0, 0)$. One has $f(0, 0, 0) = 0$.

The discriminant (also called Hessian determinant) $D = f_{xx}f_{yy} - f_{xy}^2$ at $(0,0,0)$ is $D = 7 > 0$ and $f_{xx} > 0$ so that $(0,0,0)$ is a local minimum.

b) To get the extrema on the boundary $g(x,y,z) = 2x^2+2y^2+z^2-1 = 0$ we solve the Lagrange equations $\nabla f = \lambda \nabla g, g = 0$. They are

$$\begin{aligned} 4x + 3y &= \lambda 4x \\ 3x + 4y &= \lambda 4y \\ 2z &= \lambda 2z \\ 2x^2 + 2y^2 + z^2 &= 1 \end{aligned}$$

We obtain $z = 0, x = \pm y$ or $z = \pm 1, x = y = 0$ giving 6 critical points
$(1/2,1/2,0), (-1/2,1/2,0), (1/2,-1/2,0), (-1/2,-1/2,0), (0,0,1), (0,0,-1)$.

c) Comparing the values $f(0,0,0) = 0$, $f(1/2,1/2,0) = f(-1/2,-1/2,0) = 7/4$ and $f(1/2,-1/2,0) = f(-1/2,1/2,0) = 1/2$ and $f(0,0,\pm 1) = 1$ shows that $\boxed{(1/2,1/2,0) \text{ and } (-1/2,-1/2,0) \text{ are maxima}}$ and that $\boxed{(0,0,0) \text{ is the minimum.}}$

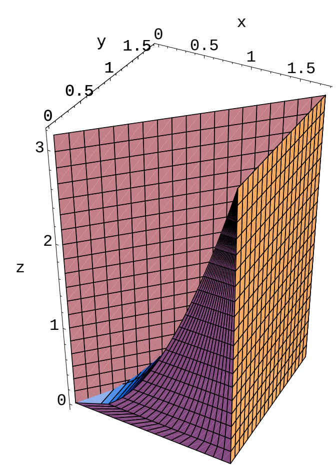
Problem 8) (10 points)

Sketch the region of integration of the following iterated integral and then evaluate the integral:

$$\int_0^\pi \left(\int_{\sqrt{z}}^{\sqrt{\pi}} \left(\int_0^x \sin(xy) dy \right) dx \right) dz \ .$$

Solution. The region is contained inside the cube $[0,\sqrt{\pi} \times [0,\sqrt{\pi}] \times [0,\pi]$. It is bounded by the surfaces $x = \sqrt{z}, x = y, z = 0, y = \sqrt{\pi}$ (see picture). The integral can not be solved in the given order. Using the picture as a guide, we write the integral as

$$\int_0^{\sqrt{\pi}} \int_0^{x^2} \int_0^x \sin(xy) \ dydzdx$$



Solve the most inner integral:

$$\int_0^{\sqrt{\pi}} \int_0^{x^2} -\frac{\cos(xy)}{x}|_0^x \ dzdx = \int_0^{\sqrt{\pi}} \int_0^{x^2} (1-\cos(x^2))/x \ dzdx$$

Now solve the z integral:

$$= \int_0^{\sqrt{\pi}} x^2(1-\cos(x^2))/x \ dx = \int_0^{\sqrt{\pi}} x(1-\cos(x^2)) \ dx$$

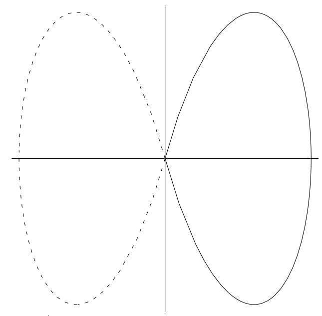
to finally get

$$= (-\frac{\sin(x^2)}{2} + \frac{x^2}{2})|_0^{\sqrt{\pi}} = \frac{\pi}{2} \ .$$

The answer is $\boxed{\frac{\pi}{2}}$.

Problem 9) (10 points)

Evaluate the line integral $\int_C \mathbf{F} \cdot dr$, where $\mathbf{F} = (x + e^x \sin(y), x + e^x \cos(y))$ and C is the right handed loop of the lemniscate described in polar coordinates as $r^2 = \cos(2\theta)$.



Solution. By Green's theorem, the integral is $\int \int_R \mathbf{curl}(F)\ dA$, where R is the region enclosed by C. From $\mathbf{F} = (P, Q) = (x + e^x \sin(y), x + e^x \cos(y))$ we calculate $\text{curl}(\mathbf{F}) = Q_x - P_y = 1 + e^x \cos(y) - \cos(y)e^x = 1$ so that the result is the area of R which is $\int_{-\pi/4}^{\pi/4} \cos(2\theta)/2\ d\theta = \frac{\sin(2\theta)}{4}\Big|_{-\pi/4}^{\pi/4} = \boxed{1/2}$.

Problem 10) (10 points)

Evaluate the line integral

$$\int_C \mathbf{F} \cdot dr\ ,$$

where C is the planar curve $\mathbf{r}(t) = (t^2, t/\sqrt{t+2}), t \in [0, 2]$ and $\mathbf{F}$ is the vector field $\mathbf{F}(x, y) = (2xy, x^2 + y)$. Do this in two different ways:

a) by verifying that $\mathbf{F}$ is conservative and replacing the path with a different path connecting $(0, 0)$ with $(4, 1)$,

b) by finding a potential U satisfying $\nabla U = \mathbf{F}$.

Solution.
a) To verify that $\mathbf{F} = (P, Q)$ is conservative, it is enough to verify that $\text{curl}(\mathbf{F}) = Q_x - P_y = 0$. This is actually the case. To calculate the line integral, we therefore can replace the path with a straight line $\gamma : \mathbf{r}(t) = (4t, t)$ and calculate $\int_\gamma \mathbf{F}\ \mathbf{dr} = \int_0^1 (8t^2, 16t^2 + t) \cdot (4, 1)\ dt = (48t^2 + t)|_0^1 = \boxed{16 + 1/2}$.

b) A potential is $f(x, y) = x^2 y + y^2/2$. The value of f at $(4, 1)$ is $16 + 1/2$. The value of f at $(0, 0)$ is 0. The difference between the potential values is $\boxed{16 + 1/2}$ again.

Problem 11) (10 points)

a) Find the line integral $\int_C \mathbf{F} \cdot dr$ of the vector field $\mathbf{F}(x, y) = (xy, x)$ along the unit circle $C : t \mapsto \mathbf{r}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$ by doing the actual line integral.

b) Find the value of the line integral obtained in a) by evaluating a double integral.

Solution.

a) $\int_0^{2\pi} (\cos(t)\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\ dt = \int_0^{2\pi} \cos^2(t) - \sin^2(t)\cos(t)\ dt = \pi + \sin^3(t)/3|_0^{2\pi} =$ $\boxed{\pi}$

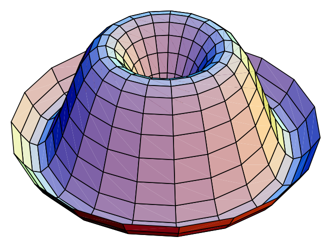
b) $\mathrm{curl}(F) = Q_x - P_y = 1 - x$. By Green's theorem, the line integral is a double integral which we evaluate using Polar coordinates
$\int\int_D (1-x)\ dA = \int_0^1 \int_0^{2\pi} (1 - \cos(\theta) r)\ d\theta dr = 2\pi/2 = \boxed{\pi}$.

SECTION SPECIFIC PROBLEMS Math 21a

Please choose one of the following problems and register your decision on the first page of this exam.

Problem 12a) (10 points)

Consider the surface given by the graph of the function $z = f(x,y) = \frac{100}{1+x^2+y^2} \sin\left(\frac{\pi}{8}(x^2+y^2)\right)$ in the region $x^2 + y^2 \leq 16$. The surface is pictured to the right.



A magnetic field $\mathbf{B}$ is given by the curl of a vector potential $\mathbf{A}$. That is, $\mathbf{B} = \nabla \times \mathbf{A} = \mathrm{curl}(A)$ and $\mathbf{A}$ is a vector field too. Suppose

$$\mathbf{A} = \left(z\sin(x^3), x\left(1 - z^2\right), \log(1 + e^{x+y+z})\right) .$$

Compute the flux of the magnetic field through this surface. (The surface has an upward pointing normal vector.)

Solution. The surface S is bounded by the curve $\gamma : \mathbf{r}(t) = (4\cos(t), 4\sin(t), 0)$. By Stokes theorem, the flux of the curl of $\mathbf{A}$ through the surface S is the line integral of A along γ:

$$\int_\gamma \mathbf{A} \cdot \mathbf{dr} = \int_0^{2\pi} (0, 4\cos(t), \log(1 + e^{x+y})) \cdot (-4\sin(t), 4\cos(t), 0)\ dt = \int_0^{2\pi} 16\cos^2(t)\ dt =$$

$\boxed{16\pi}$.

Problem 12b) (10 points)

The annual rainfall in inches in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that starting with this year, it will take over 10 years before a year occurs having a rainfall of over 50 inches.

Solution.
The probability p that no rainfall over 50 inches happens in one year is

$$p = F_{\mu,\sigma}(50) = \int_{-\infty}^{50} e^{-(x-40)^2/16}/\sqrt{\pi 16}$$

The probability that it will take more than 10 years until a rainfall over 50 inches occurs is $\boxed{p^{10}}$.

Problem 12c) (10 points)

Let S be the surface given by the equations $z = x^2 - y^2$, $x^2 + y^2 \leq 4$, with the upward pointing normal. If the vector field $\mathbf{F}$ is given by the formula $\mathbf{F}(x, y, z) = \langle -x, y, \sqrt{x^2 + y^2} \rangle$, find the flux of $\mathbf{F}$ through S.

Solution. Parameterize the surface by $\mathbf{r}(u, v) = \langle u, v, u^2 - v^2 \rangle$. Then $r_u \times r_v = \langle 1, 0, 2u \rangle \times \langle 0, 1, -2v \rangle = \langle -2u, 2v, 1 \rangle$. The flux integral is

$$\begin{aligned} \int\int_D \mathbf{F} \cdot dS &= \int\int_D \langle -u, v, \sqrt{u^2 + v^2} \rangle \cdot \langle -2u, 2v, 1 \rangle \; dxdy \\ &= \int\int_D 2(u^2 + v^2) + \sqrt{u^2 + v^2} \; dA \\ &= \int_0^{2\pi} \int_0^2 (2r^2 + r) r \; dr d\theta \\ &= 2\pi(2 \cdot 2^4/4 + 2^3/3) = 64\pi/3 \; . \end{aligned}$$

The answer is $\boxed{64\pi/3}$.

Problem 12d) (10 points)

The random variable X has a variance 2 and that the random variable Y has a variance 1. Both random variables have zero expectation. Find the correlation between $X + Y$ and $X - Y$.

Solution.

$$\begin{aligned} &\mathrm{Cov}[X + Y, X - Y] \\ &\mathrm{E}[(X + Y) - \mathrm{E}[X + Y], (X - Y) - \mathrm{E}[X - Y]\,] \\ &\mathrm{E}[(X + Y), (X - Y)\,] = \mathrm{E}[X^2 - Y^2] = \mathrm{E}[X^2] - \mathrm{E}[Y^2] \\ &\mathrm{E}[(X - \mathrm{E}[X])^2\,] - \mathrm{E}[(Y - \mathrm{E}[Y])^2\,] = \mathrm{Var}[X] - \mathrm{Var}[Y] = 2 - 1 = 1 \; , \end{aligned}$$

so that

$$\begin{aligned} &\mathrm{Corr}[X + Y, X - Y] \\ &\mathrm{Cov}[X + Y, X - Y]/(\sigma[X]\sigma[Y]) \\ &1/(\sqrt{2}) \end{aligned}$$

The solution is $\boxed{1/\sqrt{2}}$.