1/8/2003, REVIEW chapters 11 and 12 (overview/hot spots) Math 21a, O. Knill

TOPICS.

Functions of several variables.
Partial derivatives, gradient, curl, div.
Extrema of functions of two variables.
Extrema of functions with constraints.
Parameterized surfaces.
Implicit differentiation.
Double integrals.
Surface area.
Chain rule.
Directional derivatives.
Linear approximation.
Estimation using linear approximation.
Tangent planes.
Integration in Polar coordinates.
Surface integrals.
Triple integrals.
Cylindrical and spherical coordinates.

OBJECTS

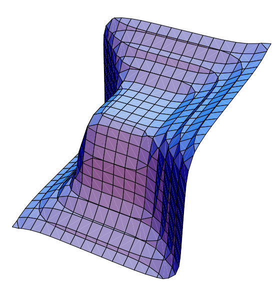
CONSTRAINED EXTREMA. Extrema of f constrained to $G = c$ are obtained with $\nabla f = \lambda \nabla g, g = c$.
CHAIN RULE. curve $r(t) = (x(t), y(t))$ and function $f(x, y)$ then $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t)$.
CHAIN RULE. $(\partial/\partial x) g(f(x, y)) = g'(f(x, y)) f_x(x, y)$, $(\partial/\partial y) g(f(x, y)) = g'(f(x, y)) f_y(x, y)$.
CRITICAL POINT. $\nabla f(x_0, y_0) = (0, 0)$, also called stationary point.
DIRECTIONAL DERIVATIVE $D_u f = \nabla f \cdot u$ (usually $||u|| = 1$ assumed).
DOUBLE INTEGRAL (I). $\int_a^b \int_{f(x)}^{g(x)} f(x, y)\; dydx$ type I integral.
DOUBLE INTEGRAL (II). $\int_a^b \int_{f(y)}^{g(y)} f(x, y)\; dxdy$ type II integral.
2D POLAR INTEGRAL. $\int \int_R f(r, \theta)\; rdrd\theta$ in polar coordinates.
3D CYLINDRICAL INTEGRAL. $\int \int \int_R f(r, \theta, z)\; rdrd\theta dz$.
3D SPHERICAL INTEGRAL. $\int \int \int_R f(\rho, \theta, \phi)\; r^2 \sin(\phi)\; drd\theta d\phi$.
GRADIENT. $f(x, y)$ function of two variables, $\nabla f(x, y) = (\partial_x f(x, y), \partial_y f(x, y)) = (f_x(x, y), f_y(x, y))$.
DISCRIMINANT. $D = f_{xx}(x, y) f_{yy}(x, y) - fxy^2(x, y)$.
IMPLICIT DIFFERENTIATION. $f(x, g(x)) = c$. $g_x = -f_x/f_y$.
LEVEL SURFACE. $f(x, y, z) = c$ has gradients $\nabla f(x, y, z)$ as normals.
LAGRANGE METHOD. $\nabla f = \lambda \nabla g, g = c$, Lagrange multiplier λ.
LINEAR APPROXIMATION. $L(x, y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0)$.
LOCAL MAXIMUM. A critical point for which $\det(H(x, y)) > 0, H_{xx}(x, y) < 0$ is a local maximum.
LOCAL MINIMUM. A critical point for which $\det(H(x, y)) > 0, H_{xx}(x, y) > 0$ is a local minimum.
PARTIAL DIFFERENTIAL EQUATION (PDE). Equation which involves partial derivatives of the function.
PDE EXAMPLE: $f_{tt} = f_{xx}$ wave equation, $f_t = f_{xx}$ heat equation.
SADDLE POINT. A critical point for which $\det(H(x, y)) < 0$.
SECOND DERIVATIVE TEST. $D < 0 \Rightarrow$ saddle, $D > 0, f_{xx} > 0 \Rightarrow$ min, $D > 0, f_{yy} < 0 \Rightarrow$ max.
TANGENT LINE. $f(x, y) = c$, $(a, b) = \nabla f(x_0, y_0)$, $d = ax_0 + by_0$. Equation $ax + by = d$.
TANGENT PLANE. $f(x, y, z) = c$, $(a, b, c) = \nabla f(x_0, y_0, z_0)$, $d = ax_0 + by_0 + cz_0$. Equation $ax + by + cz = d$.
TRIPLE INTEGRAL. $\int_a^b \int_c^d \int_e^f f(x, y, z)\; dzdydx$ example of triple integral.

CHAIN RULE. $d/dt f(g(t)) = f'(g(t)) g'(t)$ appears in different, oft disguised form $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t)$.
Compare second midterm: $d/dx f(g(x, y)) = f'(g(x, y)) g_x$ involved only the 1D chain rule.

IMPLICIT DIFFERENTIATION.

$f(x, y, z) = 1$ defines z as a function $g(x, y)$ of (x, y). The chain rule allows to compute g_x and g_y.

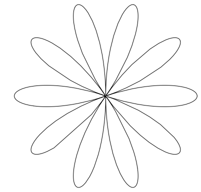
EXAMPLE: $x^5 + y^5 - z - z^7 - 1 = 0$ defines $z = g(x, y)$ and we can get $g_x(x, y) = -f_x(x, y, z)/f_z(x, y, z) = -5x^4/(1 + 7z^6)$. At the point $(1, 0, 1)$ on the surface, we would have $g_x(1, 0) = -5$.



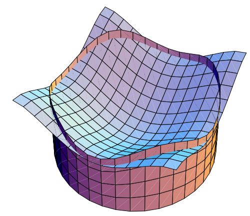
2D INTEGRATION.
Integrate $x^2 y^2$ over the triangle $x + y/2 \le 3, x > 0, y > 1$. The triangle is contained in the strip $0 \le x \le 3$. The x-integration ranges over the interval $[0, 3]$. For fixed x, we have $y \ge 1$ and $y \le 2(3 - x)$ which means that the y-bounds are $[0, 2(3 - x)]$. The double integral is $\int_0^3 \int_1^{6-2x} x^2 y^2\; dydx$.

POLAR COORDINATES. The area of a region bounded by a polar curve $r(\theta)$, where $\theta \in [a,b]$ is $\int_a^b \int_0^{r(\theta)} r \, dr d\theta = \int_a^b \frac{r^2(\theta)}{2} \, d\theta$.

EXAMPLE. The area of the **rose**: $r(\theta) = \cos(n\theta)$ is $\int_0^{2\pi} \frac{\cos^2(n\theta)}{2} \, d\theta = \frac{\pi}{2}$.



EXTREMA OF A FUNCTION. Find the maximum of $f(x,y) = 2x^2 + 2y^2 - x^4 - y^4$ on the domain $x^2 + y^2 \leq 1$.
$\nabla f(x,y) = (4x - 4x^3, 4y - 4y^3)$. The critical points inside the domain are obtained by solving $2x - 4x^3 = 0, 2y - 4y^3 = 0$ which means $x = 0, x = \pm 1$, $y = 0, y = \pm 1$. Only the point $(0,0)$ is inside the domain. The discriminant is $D = 4$ and $f_{xx} > 0$ so that $(0,0)$ is a local minimum.



SOLVING THE LAGRANGE EQUATIONS. On the boundary, we have the constraint $g(x,y) = x^2 + y^2 - 1 = 0$.

$$\begin{aligned} 4x - 4x^3 &= \lambda 2x \\ 4y - 4y^3 &= \lambda 2y \\ x^2 + y^2 &= 1 \end{aligned}$$

$$\begin{aligned} 2x(2 - \lambda - 2x^2) &= 0 \\ 2y(2 - \lambda - 2y^2) &= 0 \\ x^2 + y^2 &= 1 \end{aligned}$$

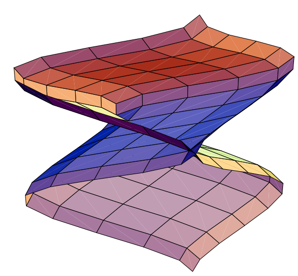
$$\begin{aligned} x = 0 \quad \text{or} \quad & x = \pm y \\ y = 0 \quad \text{or} \quad & y = \pm y \\ x^2 + y^2 \quad = \quad & 1 \end{aligned}$$

If $x = 0$, then $y = \pm 1$. If $y = 0$ then $x = \pm 1$. If $x = y$, then $x = y = \pm 1/\sqrt{2}$. If $x = -y$, then $x = -y = \pm 1/\sqrt{2}$. There are 8 critical points on the boundary. The maximum is at the four points $x = \pm 1/\sqrt{2}, y = \pm x$.

SURFACE AREA: $\int\int_R |r_u \times r_v| \, dudv$ surface area. Compare: $\int_a^b |r'(t)| \, dt$ length of curve.

EXAMPLE. The length of $r(t) = (t^2, \frac{\log(t)}{2}, 2t)$ on $t \in [1,2]$ is $\int_1^2 \sqrt{(2t)^2 + 1/(2t)^2 + 2} \, dt = \int_1^2 (2t + 1/(2t)) \, dt = t^2 + \log(t)/2|_1^2 = 3 + \log(2)/2$.

EXAMPLE. Compute the surface area of the surface $r(u,v) = (\cos(u), \cos(v), u+v)$ on $R = [0,2\pi] \times [-\pi,\pi]$.
$|r_u(u,v) \times r_v(u,v)| = |(-\sin(u), 0, 1) \times (0, -\sin(v), 1)| = |(\sin(v), \sin(u)\sin(u)\sin(v))| = \sqrt{\sin^2(u) + \sin^2(v) + \sin^2(u)\sin^2(v)}$ which can be evaluated numerically. It is very typical, that integrals appear here for which the antiderivative can not be expressed with functions like $\sin, \cos, \exp, \log$.

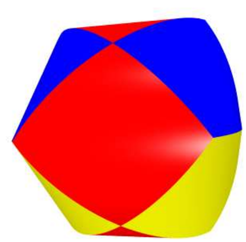


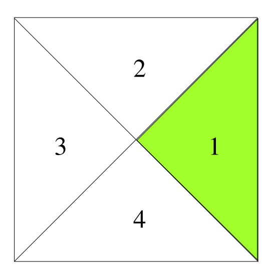
SPHERE. Volume: $\int_0^R \int_0^{2\pi} \int_0^\pi r^2 \sin(\phi) \, d\phi d\theta dr = 4R^3\pi/3$. Surface area: $\int_0^{2\pi} \int_0^\pi R^2 \sin(\phi) \, d\phi d\theta = 4\pi R^2$.

TRIPLE INTEGRAL PROBLEM: INTERSECTION OF THREE CYLINDERS. Compute the volume of the intersection of the set of points which have distance < 1 from all the three coordinate axes:

$$8 \int_{-\pi/4}^{\pi/4} \int_0^1 \sqrt{1 - r^2 \sin^2(t)} \, r \, drdt = -16/3 + 8\sqrt{2} \, .$$

The volume of one piece is the volume under the surface $f(x,y) = \sqrt{1 - y^2}$ above the triangle 1.





ADVISE: The last problem shows: a good figures often helps to solve double and triple integrals.