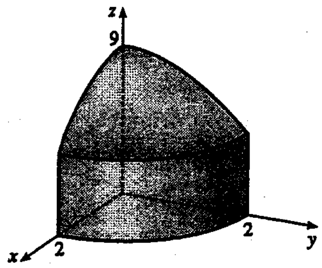
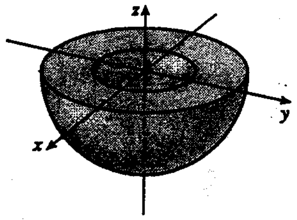
HW 10a

2. The region of integration is given in cylindrical coordinates by $E = \{(r,\theta,z) \mid 0 \le \theta \le \pi/2, 0 \le r \le 2, 0 \le z \le 9 - r^2\}$. This represents the solid region in the first octant enclosed by the circular cylinder $r = 2$, bounded above by $z = 9 - r^2$, a circular paraboloid, and bounded below by the xy-plane.



$$\begin{aligned}\int_0^{\pi/2}\int_0^2\int_0^{9-r^2} r\,dz\,dr\,d\theta &= \int_0^{\pi/2}\int_0^2 \big[rz\big]_{z=0}^{z=9-r^2}\,dr\,d\theta \\ &= \int_0^{\pi/2}\int_0^2 r(9-r^2)\,dr\,d\theta \\ &= \int_0^{\pi/2} d\theta \int_0^2 (9r - r^3)\,dr \\ &= \big[\theta\big]_0^{\pi/2}\left[\tfrac{9}{2}r^2 - \tfrac{1}{4}r^4\right]_0^2 \\ &= \tfrac{\pi}{2}(18-4) = 7\pi\end{aligned}$$

4. The region of integration is given in spherical coordinates by $E = \{(\rho,\theta,\phi) \mid 1 \le \rho \le 2, 0 \le \theta \le 2\pi, \pi/2 \le \phi \le \pi\}$. This represents the solid region between the spheres $\rho = 1$ and $\rho = 2$ and below the xy-plane.



$$\begin{aligned}\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta &= \int_0^{2\pi} d\theta \int_{\pi/2}^{\pi}\sin\phi\,d\phi \int_1^2 \rho^2\,d\rho \\ &= \big[\theta\big]_0^{2\pi}\big[-\cos\phi\big]_{\pi/2}^{\pi}\left[\tfrac{1}{3}\rho^3\right]_1^2 \\ &= 2\pi\,(1)\left(\tfrac{7}{3}\right) = \tfrac{14\pi}{3}\end{aligned}$$

10. In cylindrical coordinates, E is bounded by the cylinder $r = 1$ and the planes $z = 0$, $z = y = r\sin\theta$ with $y \ge 0$ $\Rightarrow$ $0 \le \theta \le \pi$, so E is given by $\{(r,\theta,z) \mid 0 \le \theta \le \pi, 0 \le r \le 1, 0 \le z \le r\sin\theta\}$. Thus

$$\begin{aligned}\iiint_E xz\,dV &= \int_0^{\pi}\int_0^1\int_0^{r\sin\theta} r^2 z\cos\theta\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^1 \left[\tfrac{1}{2}z^2\right]_{z=0}^{z=r\sin\theta} r^2\cos\theta\,dr\,d\theta \\ &= \tfrac{1}{2}\int_0^{\pi}\int_0^1 r^4\sin^2\theta\cos\theta\,dr\,d\theta = \tfrac{1}{2}\int_0^{\pi}\left[\tfrac{1}{5}r^5\right]_{r=0}^{r=1}\sin^2\theta\cos\theta\,d\theta \\ &= \tfrac{1}{10}\int_0^{\pi}(\sin^2\theta\cos\theta)\,d\theta = \tfrac{1}{30}\sin^3\theta\Big]_0^{\pi} = 0\end{aligned}$$

16. In spherical coordinates, H is represented by $\{(\rho,\theta,\phi) \mid 0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}\}$. Thus

$$\begin{aligned}\iiint_H (x^2+y^2)\,dV &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (\rho^2\sin^2\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi} d\theta\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^1\rho^4\,d\rho \\ &= \big[\theta\big]_0^{2\pi}\left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi/2}\left[\tfrac{1}{5}\rho^5\right]_0^1 = \tfrac{4\pi}{15}\end{aligned}$$

30. The region of integration E is the region above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 18$ in the first octant. Because E is in the first octant we have $0 \le \theta \le \frac{\pi}{2}$. The cone has equation $\phi = \frac{\pi}{4}$ (as in Example 4) and so $0 \le \phi \le \frac{\pi}{4}$. Also $0 \le \rho \le \sqrt{18} = 3\sqrt{2}$. So the integral becomes

$$\begin{aligned}\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{3\sqrt2} \rho^4\sin\phi\,d\rho\,d\phi\,d\theta &= \int_0^{\pi/2} d\theta\int_0^{\pi/4}\sin\phi\,d\phi\int_0^{3\sqrt2}\rho^4\,d\rho \\ &= \big[\theta\big]_0^{\pi/2}\big[-\cos\phi\big]_0^{\pi/4}\left[\tfrac{1}{5}\rho^5\right]_0^{3\sqrt2} \\ &= \left(\tfrac{\pi}{2}\right)\left(1-\tfrac{\sqrt2}{2}\right)\left(\tfrac{972\sqrt2}{5}\right) = 486\pi\left(\tfrac{\sqrt2-1}{5}\right)\end{aligned}$$