HW11a

16. (a) $f_x(x,y,z) = 2xz + y^2$ implies $f(x,y,z) = x^2z + xy^2 + g(y,z)$ and so $f_y(x,y,z) = 2xy + g_y(y,z)$. But $f_y(x,y,z) = 2xy$ so $g_y(y,z) = 0 \quad \Rightarrow \quad g(y,z) = h(z)$. Thus $f(x,y,z) = x^2z + xy^2 + h(z)$ and $f_z(x,y,z) = x^2 + h'(z)$. But $f_z(x,y,z) = x^2 + 3z^2$, so $h'(z) = 3z^2 \quad \Rightarrow \quad h(z) = z^3 + K$. Hence $f(x,y,z) = x^2z + xy^2 + z^3$ (taking $K = 0$).

(b) $t = 0$ corresponds to the point $(0,1,-1)$ and $t = 1$ corresponds to $(1,2,1)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,2,1) - f(0,1,-1) = 6 - (-1) = 7.$

26. $\nabla f(x,y) = \cos(x-2y)\,\mathbf{i} - 2\cos(x-2y)\,\mathbf{j}$

(a) We use Theorem 2: $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where C_1 starts at $t = a$ and ends at $t = b$. So because $f(0,0) = \sin 0 = 0$ and $f(\pi,\pi) = \sin(\pi - 2\pi) = 0$, one possible curve C_1 is the straight line from $(0,0)$ to (π,π); that is, $\mathbf{r}(t) = \pi t\,\mathbf{i} + \pi t\,\mathbf{j}$, $0 \le t \le 1$.

(b) From (a), $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. So because $f(0,0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2},0\right) = 1$, one possible curve C_2 is $\mathbf{r}(t) = \frac{\pi}{2}t\,\mathbf{i}$, $0 \le t \le 1$, the straight line from $(0,0)$ to $\left(\frac{\pi}{2},0\right)$.

8. The region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 1, 3x \le y \le 3\}$, so

$$\int_C x^2y^2\,dx + 4xy^3\,dy = \iint_D \left[\frac{\partial}{\partial x}\left(4xy^3\right) - \frac{\partial}{\partial y}\left(x^2y^2\right)\right] dA = \int_0^1 \int_{3x}^3 \left(4y^3 - 2x^2y\right) dy\,dx$$

$$= \int_0^1 \left[y^4 - x^2y^2\right]_{y=3x}^{y=3} dx = \int_0^1 \left(81 - 9x^2 - 72x^4\right) dx = 81 - 3 - \tfrac{72}{5} = \tfrac{318}{5}$$

18. By Green's Theorem, $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x\,dx + (x^3 + 3xy^2)\,dy = \iint_D (3x^2 + 3y^2 - 0)\,dA$, where D is the semicircular region bounded by C. Converting to polar coordinates, we have $W = 3\int_0^2 \int_0^\pi r^2 \cdot r\,d\theta\,dr = 3\pi\left[\frac{1}{4}r^4\right]_0^2 = 12\pi$.

20. $A = \oint_C x\,dy = \int_0^{2\pi} (\cos t)\left(3\sin^2 t\cos t\right) dt = 3\int_0^{2\pi} \frac{1}{8}\left(1 - \cos 4t\right) dt = \frac{3}{4}\pi$