HW 11b

4. (a) curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & xe^y & ye^z \end{vmatrix} = (e^z - 0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (e^y - 0)\,\mathbf{k} = e^z\,\mathbf{i} + e^y\,\mathbf{k}$

(b) div $\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(0) + \dfrac{\partial}{\partial y}(xe^y) + \dfrac{\partial}{\partial z}(ye^z) = xe^y + ye^z$

6. (a) curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{x^2+y^2+z^2} & \dfrac{y}{x^2+y^2+z^2} & \dfrac{z}{x^2+y^2+z^2} \end{vmatrix}$

$$= \frac{1}{(x^2+y^2+z^2)^2}\left[(-2yz+2yz)\,\mathbf{i} - (-2xz+2xz)\,\mathbf{j} + (-2xy+2xy)\,\mathbf{k}\right] = \mathbf{0}$$

(b) div $\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2+y^2+z^2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2+y^2+z^2}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{z}{x^2+y^2+z^2}\right)$

$$= \frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2-2y^2}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2-2z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2}$$

10. (a) curl $f = \nabla \times f$ is meaningless because f is a scalar field.

(b) grad f is a vector field.

(c) div $\mathbf{F}$ is a scalar field.

(d) curl (grad f) is a vector field.

(e) grad $\mathbf{F}$ is meaningless because $\mathbf{F}$ is not a scalar field.

(f) grad (div $\mathbf{F}$) is a vector field.

(g) div(grad f) is a scalar field.

(h) grad (div f) is meaningless because f is a scalar field.

(i) curl (curl $\mathbf{F}$) is a vector field.

(j) div(div $\mathbf{F}$) is meaningless because div $\mathbf{F}$ is a scalar field.

(k) (grad f) $\times$ (div $\mathbf{F}$) is meaningless because div $\mathbf{F}$ is a scalar field.

(l) div(curl (grad f)) is a scalar field.

12. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$, $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = x$ implies $f(x,y,z) = \frac{1}{2}x^2 + g(y,z)$ and $f_y(x,y,z) = g_y(y,z)$. But $f_y(x,y,z) = y$, so $g(y,z) = \frac{1}{2}y^2 + h(z)$ and $f(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + h(z)$. Thus $f_z(x,y,z) = h'(z)$ but $f_z(x,y,z) = z$ so $h(z) = \frac{1}{2}z^2 + K$ and $f(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 + K$.

34. Let $\mathbf{H} = \langle h_1, h_2, h_3 \rangle$ and $\mathbf{E} = \langle E_1, E_2, E_3 \rangle$.

(a) $\nabla \times (\nabla \times \mathbf{E}) = \nabla \times (\text{curl}\,\mathbf{E}) = \nabla \times \left(-\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t}\right) = -\dfrac{1}{c}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial h_1/\partial t & \partial h_2/\partial t & \partial h_3/\partial t \end{vmatrix}$

$$= -\frac{1}{c}\left[\left(\frac{\partial^2 h_3}{\partial y\,\partial t} - \frac{\partial^2 h_2}{\partial z\,\partial t}\right)\mathbf{i} + \left(\frac{\partial^2 h_1}{\partial z\,\partial t} - \frac{\partial^2 h_3}{\partial x\,\partial t}\right)\mathbf{j} + \left(\frac{\partial^2 h_2}{\partial x\,\partial t} - \frac{\partial^2 h_1}{\partial y\,\partial t}\right)\mathbf{k}\right]$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right)\mathbf{k}\right]$$

(assuming that the partial derivatives are continuous so that the order of differentiation does not matter)

$$= -\frac{1}{c}\frac{\partial}{\partial t}\,\text{curl}\,\mathbf{H} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$