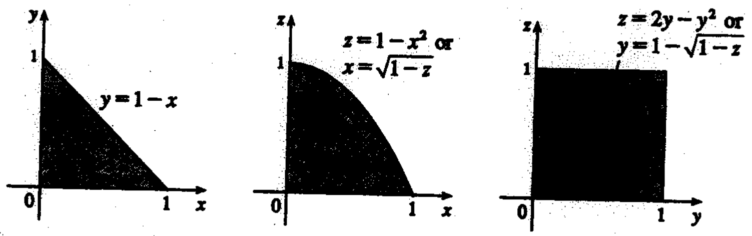
HW9

4. $\int_0^1 \int_x^{2x} \int_0^y 2xyz\,dz\,dy\,dx = \int_0^1 \int_x^{2x} \left[xyz^2\right]_{z=0}^{z=y} dy\,dx = \int_0^1 \int_x^{2x} xy^3\,dy\,dx$

$= \int_0^1 \left[\frac{1}{4}xy^4\right]_{y=x}^{y=2x} dx = \int_0^1 \frac{15}{4}x^5\,dx = \left[\frac{5}{8}x^6\right]_0^1 = \frac{5}{8}$

30.



The projections of E onto the xy- and xz-planes are as in the first two diagrams and so

$$\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,dz\,dx = \int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\,dy\,dx\,dz$$
$$= \int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx$$

Now the surface $z = 1 - x^2$ intersects the plane $y = 1 - x$ in a curve whose projection in the yz-plane is $z = 1 - (1-y)^2$ or $z = 2y - y^2$. So we must split up the projection of E on the yz-plane into two regions as in the third diagram. For (y, z) in R_1, $0 \le x \le 1 - y$ and for (y, z) in R_2, $0 \le x \le \sqrt{1-z}$, and so the given integral is also equal to

$$\int_0^1\int_0^{1-\sqrt{1-z}}\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz + \int_0^1\int_{1-\sqrt{1-z}}^1\int_0^{1-y} f(x,y,z)\,dx\,dy\,dz$$
$$= \int_0^1\int_0^{2y-y^2}\int_0^{1-y} f(x,y,z)\,dx\,dz\,dy + \int_0^1\int_{2y-y^2}^1\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy.$$

34. $m = \int_{-1}^1\int_0^{1-y^2}\int_0^{1-z} 4\,dx\,dz\,dy = 4\int_{-1}^1\int_0^{1-y^2}(1-z)\,dz\,dy = 4\int_{-1}^1\left[z - \frac{1}{2}z^2\right]_{z=0}^{z=1-y^2} dy$

$= 2\int_{-1}^1 (1-y^4)\,dy = \frac{16}{5}$,

$M_{yz} = \int_{-1}^1\int_0^{1-y^2}\int_0^{1-z} 4x\,dx\,dz\,dy = 2\int_{-1}^1\int_0^{1-y^2}(1-z)^2\,dz\,dy = 2\int_{-1}^1\left[-\frac{1}{3}(1-z)^3\right]_{z=0}^{z=1-y^2} dy$

$= \frac{2}{3}\int_{-1}^1 (1-y^6)\,dy = \left(\frac{4}{3}\right)\left(\frac{6}{7}\right) = \frac{24}{21} = \frac{8}{7}$

$M_{xz} = \int_{-1}^1\int_0^{1-y^2}\int_0^{1-z} 4y\,dx\,dz\,dy = \int_{-1}^1\int_0^{1-y^2} 4y(1-z)\,dz\,dy$

$= \int_{-1}^1\left[4y(1-y^2) - 2y(1-y^2)^2\right] dy = \int_{-1}^1 (2y - 2y^5)\,dy = 0$ (the integrand is odd)

$M_{xy} = \int_{-1}^1\int_0^{1-y^2}\int_0^{1-z} 4z\,dx\,dz\,dy = \int_{-1}^1\int_0^{1-y^2}(4z - 4z^2)\,dz\,dy$

$= 2\int_{-1}^1\left[(1-y^2)^2 - \frac{2}{3}(1-y^2)^3\right] dy = 2\int_{-1}^1\left[\frac{1}{3} - y^4 + \frac{2}{3}y^6\right] dy$

$= \left[\frac{4}{3}y - \frac{4}{5}y^5 + \frac{8}{21}y^7\right]_0^1 = \frac{96}{105} = \frac{32}{35}$

Thus, $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{5}{14}, 0, \frac{2}{7}\right)$

8. $\iiint_E yz\cos(x^5)\,dV = \int_0^1\int_0^x\int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx = \int_0^1\int_0^x\left[\frac{1}{2}yz^2\cos(x^5)\right]_{z=x}^{z=2x} dy\,dx$

$= \frac{1}{2}\int_0^1\int_0^x 3x^2y\cos(x^5)\,dy\,dx = \frac{1}{2}\int_0^1\left[\frac{3}{2}x^2y^2\cos(x^5)\right]_{y=0}^{y=x} dx$

$= \frac{3}{4}\int_0^1 x^4\cos(x^5)\,dx = \frac{3}{4}\left[\frac{1}{5}\sin(x^5)\right]_0^1 = \frac{3}{20}(\sin 1 - \sin 0) = \frac{3}{20}\sin 1$

6. $\int_0^1\int_0^z\int_0^y ze^{-y^2}\,dx\,dy\,dz = \int_0^1\int_0^z\left[xze^{-y^2}\right]_{x=0}^{x=y} dy\,dz = \int_0^1\int_0^z yze^{-y^2}\,dy\,dz = \int_0^1\left[-\frac{1}{2}ze^{-y^2}\right]_{y=0}^{y=z} dz$

$= \int_0^1 -\frac{1}{2}z\left(e^{-z^2} - 1\right) dz = \frac{1}{2}\int_0^1\left(z - ze^{-z^2}\right) dz$

$= \frac{1}{2}\left[\frac{1}{2}z^2 + \frac{1}{2}e^{-z^2}\right]_0^1 = \frac{1}{4}\left(1 + e^{-1} - 0 - 1\right) = \frac{1}{4e}$