8. $f_x(2,1)$ is the rate of change of f at $(2,1)$ in the x-direction. If we start at $(2,1)$, where $f(2,1) = 10$, and move in the positive x-direction, we reach the next contour line (where $f(x,y) = 12$) after approximately 0.6 units. This represents an average rate of change of about $\frac{2}{0.6}$. If we approach the point $(2,1)$ from the left (moving in the positive x-direction) the output values increase from 8 to 10 with an increase in x of approximately 0.9 units, corresponding to an average rate of change of $\frac{2}{0.9}$. A good estimate for $f_x(2,1)$ would be the average of these two, so $f_x(2,1) \approx 2.8$. Similarly, $f_y(2,1)$ is the rate of change of f at $(2,1)$ in the y-direction. If we approach $(2,1)$ from below, the output values decrease from 12 to 10 with a change in y of approximately 1 unit, corresponding to an average rate of change of -2. If we start at $(2,1)$ and move in the positive y-direction, the output values decrease from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{-2}{0.9}$. Averaging these two results, we estimate $f_y(2,1) \approx -2.1$.

14. $f(x,y) = x^5 + 3x^3y^2 + 3xy^4 \Rightarrow f_x(x,y) = 5x^4 + 3 \cdot 3x^2 \cdot y^2 + 3 \cdot 1 \cdot y^4 = 5x^4 + 9x^2y^2 + 3y^4$,
$f_y(x,y) = 0 + 3x^3 \cdot 2y + 3x \cdot 4y^3 = 6x^3y + 12xy^3$.

26. $f(x,y,z) = x^2 e^{yz} \Rightarrow f_x(x,y,z) = 2xe^{yz}$, $f_y(x,y,z) = x^2 e^{yz}(z) = x^2 z e^{yz}$,
$f_z(x,y,z) = x^2 e^{yz}(y) = x^2 y e^{yz}$.

48. $f(x,y) = \ln(3x+5y) \Rightarrow f_x(x,y) = \dfrac{3}{3x+5y}$, $f_y(x,y) = \dfrac{5}{3x+5y}$. Then
$f_{xx}(x,y) = 3(-1)(3x+5y)^{-2}(3) = -\dfrac{9}{(3x+5y)^2}$, $f_{xy}(x,y) = -\dfrac{15}{(3x+5y)^2}$, $f_{yx}(x,y) = -\dfrac{15}{(3x+5y)^2}$,
and $f_{yy}(x,y) = -\dfrac{25}{(3x+5y)^2}$.

78. $f_x(0,0) = \lim\limits_{h \to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{(h^3+0)^{1/3} - 0}{h} = \lim\limits_{h \to 0} \dfrac{h}{h} = 1$.
Or: Let $g(x) = f(x,0) = \sqrt[3]{x^3 + 0} = x$. Then $g'(x) = 1$ and $g'(0) = 1$ so, by (1), $f_x(0,0) = g'(0) = 1$.

4. $z = f(x,y) = y \ln x \Rightarrow f_x(x,y) = y/x$, $f_y(x,y) = \ln x$, so $f_x(1,4) = 4$, $f_y(1,4) = 0$, and an equation of the tangent plane is $z - 0 = f_x(1,4)(x-1) + f_y(1,4)(y-4) \Rightarrow z = 4(x-1) + 0(y-4)$ or $z = 4x - 4$.

16. From the table, $f(40,20) = 28$. To estimate $f_v(40,20)$ and $f_t(40,20)$ we follow the procedure used in Exercise 11.3.4. Since $f_v(40,20) = \lim\limits_{h \to 0} \dfrac{f(40+h,20) - f(40,20)}{h}$, we approximate this quantity with $h = \pm 10$ and use the values given in the table: $f_v(40,20) \approx \dfrac{f(50,20) - f(40,20)}{10} = \dfrac{40 - 28}{10} = 1.2$,
$f_v(40,20) \approx \dfrac{f(30,20) - f(40,20)}{-10} = \dfrac{17 - 28}{-10} = 1.1$. Averaging these values gives $f_v(40,20) \approx 1.15$.
Similarly, $f_t(40,20) = \lim\limits_{h \to 0} \dfrac{f(40,20) - f(40,20)}{h}$, so we use $h = 10$ and $h = -5$:
$f_t(40,20) \approx \dfrac{f(40,30) - f(40,20)}{10} = \dfrac{31 - 28}{10} = 0.3$, $f_t(40,20) \approx \dfrac{f(40,15) - f(40,20)}{-5} = \dfrac{25 - 28}{-5} = 0.6$.
Averaging these values gives $f_t(40,15) \approx 0.45$. The linear approximation, then, is
$$f(v,t) \approx f(40,20) + f_v(40,20)(v - 40) + f_t(40,20)(t - 20)$$
$$\approx 28 + 1.15(v - 40) + 0.45(t - 20)$$

40. (a) $\lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{0 - 0}{h} = 0$ and $\lim\limits_{h \to 0} \dfrac{f(0,h) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{0 - 0}{h} = 0$. Thus $f_x(0,0) = f_y(0,0) = 0$. To show that f isn't differentiable at $(0,0)$ we need only show that f is not continuous at $(0,0)$ and apply Exercise 39. As $(x,y) \to (0,0)$ along the x-axis $f(x,y) = 0/x^2 = 0$ for $x \neq 0$ so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But as $(x,y) \to (0,0)$ along the line $y = x$, $f(x,x) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$ so $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along this line. Thus $\lim\limits_{(x,y) \to (0,0)} f(x,y)$ doesn't exist, so f is discontinuous at $(0,0)$ and thus not differentiable there.

30. Here $dV = \Delta V = 0.3$, $dT = \Delta T = -5$, $P = 8.31\dfrac{T}{V}$, so
$dP = \left(\dfrac{8.31}{V}\right)dT - \dfrac{8.31 \cdot T}{V^2}dV = 8.31\left[-\dfrac{5}{12} - \dfrac{310}{144} \cdot \dfrac{3}{10}\right] \approx -8.83$.
Thus the pressure will drop by about 8.83 kPa.

12. $f(x,y) = \sin(2x+3y)$. The partial derivatives are $f_x(x,y) = 2\cos(2x+3y)$ and $f_y(x,y) = 3\cos(2x+3y)$, so $f_x(-3,2) = 2$ and $f_y(-3,2) = 3$. Both f_x and f_y are continuous functions, so f is differentiable at $(-3,2)$, and the linearization of f at $(-3,2)$ is
$L(x,y) = f(-3,2) + f_x(-3,2)(x+3) + f_y(-3,2)(y-2) = 0 + 2(x+3) + 3(y-2) = 2x + 3y$.