6. $f(x,y) = x^2 + y^2$, $g(x,y) = x^4 + y^4 = 1 \;\Rightarrow\; \nabla f = \langle 2x, 2y\rangle$, $\lambda\nabla g = \langle 4\lambda x^3, 4\lambda y^3\rangle$. Then $x = 2\lambda x^3$ implies $x = 0$ or $\lambda = \dfrac{1}{2x^2}$. If $x = 0$, then $x^4 + y^4 = 1$ implies $y = \pm 1$. But $y = 2\lambda y^3$ implies $y = 0$ so $x = \pm 1$ or $\lambda = \dfrac{1}{2y^2}$ and $x^2 = y^2$ and $2x^4 = 1$ so $x = \pm\frac{1}{\sqrt[4]{2}}$. Hence the possible points are $(0, \pm 1)$, $(\pm 1, 0)$, $\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right)$, with the maximum value of f on $x^4 + y^4 = 1$ being $f\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$ and the minimum value being $f(0, \pm 1) = f(\pm 1, 0) = 1$.

8. $f(x,y,z) = 8x - 4z$, $g(x,y,z) = x^2 + 10y^2 + z^2 = 5 \;\Rightarrow\; \nabla f = \langle 8, 0, -4\rangle$, $\lambda\nabla g = \langle 2\lambda x, 20\lambda y, 2\lambda z\rangle$. Then $2\lambda x = 8$, $20\lambda y = 0$, $2\lambda z = -4$ imply $x = \dfrac{4}{\lambda}$, $y = 0$, and $z = -\dfrac{2}{\lambda}$. But $5 = x^2 + 10y^2 + z^2 = \left(\dfrac{4}{\lambda}\right)^2 + 10\,(0)^2 + \left(-\dfrac{2}{\lambda}\right)^2 \;\Rightarrow\; 5 = \dfrac{20}{\lambda^2} \;\Rightarrow\; \lambda = \pm 2$, so f has possible extreme values at the points $(2, 0, -1)$, $(-2, 0, 1)$. The maximum of f on $x^2 + 10y^2 + z^2 = 5$ is $f(2, 0, -1) = 20$, and the minimum is $f(-2, 0, 1) = -20$.

10. $f(x,y,z) = x^2y^2z^2$, $g(x,y,z) = x^2 + y^2 + z^2 = 1 \;\Rightarrow\; \nabla f = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $\nabla f = \lambda\nabla g$ implies (1) $\lambda = y^2z^2 = x^2z^2 = x^2y^2$ and $\lambda \neq 0$, or (2) $\lambda = 0$ and one or two (but not three) of the coordinates are 0. If (1) then $x^2 = y^2 = z^2 = \frac{1}{3}$. The minimum value of f on the sphere occurs in case (2) with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from (1), that is, the points $\left(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

12. $f(x,y,z) = x^4 + y^4 + z^4$, $g(x,y,z) = x^2 + y^2 + z^2 = 1 \;\Rightarrow\; \nabla f = \langle 4x^3, 4y^3, 4z^3\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$.

Case 1: If $x \neq 0$, $y \neq 0$ and $z \neq 0$ then $\nabla f = \lambda\nabla g$ implies $\lambda = 2x^2 = 2y^2 = 2z^2$ or $x^2 = y^2 = z^2 = \frac{1}{3}$ yielding 8 points each with an f-value of $\frac{1}{3}$.

Case 2: If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f-value is $\frac{1}{2}$.

Case 3: If exactly two of the variables are 0, then the third variable has value ± 1 with corresponding f-value of 1.

Thus on $x^2 + y^2 + z^2 = 1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.

18. $f(x,y) = 2x^2 + 3y^2 - 4x - 5 \;\Rightarrow\; \nabla f = \langle 4x - 4, 6y\rangle = \langle 0, 0\rangle \;\Rightarrow\; x = 1, y = 0$. Thus $(1, 0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x,y) = x^2 + y^2 = 16 \;\Rightarrow\; \lambda\nabla g = \langle 2\lambda x, 2\lambda y\rangle$, so $6y = 2\lambda y \;\Rightarrow\;$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \;\Rightarrow\; x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1, 0) = -7$, $f(4, 0) = 11$, $f(-4, 0) = 43$, and $f(-2, \pm 2\sqrt{3}) = 47$. Thus the maximum value of $f(x,y)$ on the disk $x^2 + y^2 \le 16$ is $f(-2, \pm 2\sqrt{3}) = 47$, and the minimum value is $f(1, 0) = -7$.

19. $f(x,y) = e^{-xy}$. For the interior of the region, we find the critical points: $f_x = -ye^{-xy}$, $f_y = -xe^{-xy}$, so the only critical point is $(0, 0)$, and $f(0, 0) = 1$. For the boundary, we use Lagrange multipliers. $g(x,y) = x^2 + 4y^2 = 1 \;\Rightarrow\; \lambda\nabla g = \langle 2\lambda x, 8\lambda y\rangle$, so setting $\nabla f = \lambda\nabla g$ we get $-ye^{-xy} = 2\lambda x$ and $-xe^{-xy} = 8\lambda y$. The first of these gives $e^{-xy} = -2\lambda x/y$, and then the second gives $-x(-2\lambda x/y) = 8\lambda y \;\Rightarrow\; x^2 = 4y^2$. Solving this last equation with the constraint $x^2 + 4y^2 = 1$ gives $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{2\sqrt{2}}$. Now $f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{1/4} \approx 1.284$ and $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-1/4} \approx 0.779$. The former are the maxima on the region and the latter are the minima.

24. Let $f(x,y,z) = s(s-x)(s-y)(s-z)$, $g(x,y,z) = x + y + z$. Then $\nabla f = \langle -s(s-y)(s-z), -s(s-x)(s-z), -s(s-x)(s-y)\rangle$, $\lambda\nabla g = \langle \lambda, \lambda, \lambda\rangle$. Thus (1) $(s-y)(s-z) = (s-x)(s-z)$ and (2) $(s-x)(s-z) = (s-x)(s-y)$. (1) implies $x = y$ while (2) implies $y = z$, so $x = y = z = p/3$ and the triangle with maximum area is equilateral.

36. Let the dimensions of the box be x, y, and z, so its volume is $f(x,y,z) = xyz$, its surface area is $g(x,y,z) = xy + yz + xz = 750$ and its total edge length is $h(x,y,z) = x + y + z = 50$. Then $\nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda(y+z), \lambda(x+z), \lambda(x+y) \rangle + \langle \mu, \mu, \mu \rangle$. So (1) $yz = \lambda(y+z) + \mu$, (2) $xz = \lambda(x+z) + \mu$, and (3) $xy = \lambda(x+y) + \mu$. Notice that the box can't be a cube or else $x = y = z = \frac{50}{3}$ but then $xy + yz + xz = \frac{2500}{3} \neq 750$. Assume x is the distinct side, that is, $x \neq y$, $x \neq z$. Then (1) minus (2) implies $z\,(y - x) = \lambda(y - x)$ or $\lambda = z$, and (1) minus (3) implies $y(z - x) = \lambda(z - x)$ or $\lambda = y$. So $y = z = \lambda$ and $x + y + z = 50$ implies $x = 50 - 2\lambda$; also $xy + yz + xz = 750$ implies $x(2\lambda) + \lambda^2 = 750$. Hence

$$50 - 2\lambda = \frac{750 - \lambda^2}{2\lambda} \text{ or } 3\lambda^2 - 100\lambda + 750 = 0 \text{ and } \lambda = \frac{50 \pm 5\sqrt{10}}{3},$$ giving the points

$\left(\frac{1}{3}\left(50 \mp 10\sqrt{10}\right), \frac{1}{3}\left(50 \pm 5\sqrt{10}\right), \frac{1}{3}\left(50 \pm 5\sqrt{10}\right)\right)$. Thus the minimum of f is

$f\left(\frac{1}{3}\left(50 - 10\sqrt{3}\right), \frac{1}{3}\left(50 + 5\sqrt{10}\right), \frac{1}{3}\left(50 + 5\sqrt{10}\right)\right) = \frac{1}{27}\left(87{,}500 - 2500\sqrt{10}\right)$, and its maximum is

$f\left(\frac{1}{3}\left(50 + 10\sqrt{10}\right), \frac{1}{3}\left(50 - 5\sqrt{10}\right), \frac{1}{3}\left(50 - 5\sqrt{10}\right)\right) = \frac{1}{27}\left(87{,}500 + 2500\sqrt{10}\right)$.

Note: If either y or z is the distinct side, then symmetry gives the same result.

42. (a) Let $f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} x_i y_i$, $g(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$, and $h(x_1, \ldots, x_n) = \sum_{i=1}^{n} y_i^2$. Then $\nabla f = \nabla \sum_{i=1}^{n} x_i y_i = \langle y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_n \rangle$, $\nabla g = \nabla \sum_{i=1}^{n} x_i^2 = \langle 2x_1, 2x_2, \ldots, 2x_n, 0, 0, \ldots, 0 \rangle$ and $\nabla h = \nabla \sum_{i=1}^{n} y_i^2 = \langle 0, 0, \ldots, 0, 2y_1, 2y_2, \ldots, 2y_n \rangle$. So $\nabla f = \lambda \nabla g + \mu \nabla h \;\Leftrightarrow\; y_i = 2\lambda x_i$ and $x_i = 2\mu y_i$, $1 \le i \le n$. Then $1 = \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} 4\lambda^2 x_i^2 = 4\lambda^2 \sum_{i=1}^{n} x_i^2 = 4\lambda^2 \;\Rightarrow\; \lambda = \pm\frac{1}{2}$. If $\lambda = \frac{1}{2}$ then $y_i = 2\left(\frac{1}{2}\right)x_i = x_i$, $1 \le i \le n$. Thus $\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} x_i^2 = 1$. Similarly if $\lambda = -\frac{1}{2}$ we get $y_i = -x_i$ and $\sum_{i=1}^{n} x_i y_i = -1$. Similarly we get $\mu = \pm\frac{1}{2}$ giving $y_i = \pm x_i$, $1 \le i \le n$, and $\sum_{i=1}^{n} x_i y_i = \pm 1$.

Thus the maximum value of $\sum_{i=1}^{n} x_i y_i$ is 1

(b) Here we assume $\sum_{i=1}^{n} a_i^2 \neq 0$ and $\sum_{i=1}^{n} b_i^2 \neq 0$. (If $\sum_{i=1}^{n} a_i^2 = 0$, then each $a_i = 0$ and so the inequality is trivially true.) $x_i = \dfrac{a_i}{\sqrt{\sum a_i^2}} \;\Rightarrow\; \sum x_i^2 = \dfrac{\sum a_i^2}{\sum a_i^2} = 1$, and $y_i = \dfrac{b_i}{\sqrt{\sum b_i^2}} \;\Rightarrow\; \sum y_i^2 = \dfrac{\sum b_i^2}{\sum b_i^2} = 1$. Therefore, from (a), $\sum x_i y_i = \sum \dfrac{a_i b_i}{\sqrt{\sum a_i^2}\sqrt{\sum b_i^2}} \le 1 \;\Leftrightarrow\; \sum a_i b_i \le \sqrt{\sum a_i^2}\sqrt{\sum b_i^2}$.