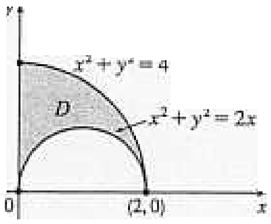
14.



$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x\ge 0,\,y\ge 0}} x\,dA - \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\tfrac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\tfrac{1}{3}\left(8\cos^4\theta\right)d\theta$$

$$= \tfrac{8}{3} - \tfrac{8}{12}\left[\cos^3\theta\sin\theta + \tfrac{3}{2}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2}$$

$$= \tfrac{8}{3} - \tfrac{2}{3}\left[0 + \tfrac{3}{2}\left(\tfrac{\pi}{2}\right)\right] = \tfrac{16-3\pi}{6}$$

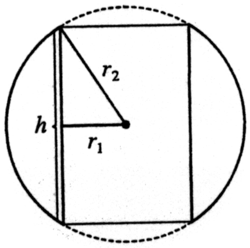
22. (a) Here the region in the xy-plane is the annular region $r_1^2 \le x^2+y^2 \le r_2^2$ and the desired volume is twice that above the xy-plane. Hence

$$V = 2\iint_{r_1^2\le x^2+y^2\le r_2^2}\sqrt{r_2^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_{r_1}^{r_2}\sqrt{r_2^2-r^2}\,r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}d\theta\int_{r_1}^{r_2}\sqrt{r_2^2-r^2}\,r\,dr = \tfrac{4\pi}{3}\left[-\left(r_2^2-r^2\right)^{3/2}\right]_{r_1}^{r_2} = \tfrac{4\pi}{3}\left(r_2^2-r_1^2\right)^{3/2}$$

(b) A cross-sectional cut is shown in the figure. So $r_2^2 = \left(\tfrac{1}{2}h\right)^2 + r_1^2$ or $\tfrac{1}{4}h^2 = r_2^2 - r_1^2$. Thus the volume in terms of h is

$V = \tfrac{4\pi}{3}\left(\tfrac{1}{4}h^2\right)^{3/2} = \tfrac{\pi}{6}h^3.$



30. (a) The total amount of water supplied each hour to the region within R feet of the sprinkler is

$$V = \int_0^{2\pi}\int_0^R e^{-r}r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^R re^{-r}\,dr = [\theta]_0^{2\pi}\left[-re^{-r}-e^{-r}\right]_0^R$$

$$= 2\pi\left[-Re^{-R}-e^{-R}+0+1\right] = 2\pi\left(1-Re^{-R}-e^{-R}\right)\ \text{ft}^3$$

(b) The average amount of water per hour per square foot supplied to the region within R feet of the sprinkler is $\dfrac{V}{\text{area of region}} = \dfrac{V}{\pi R^2} = \dfrac{2\left(1-Re^{-R}-e^{-R}\right)}{R^2}$ ft^3 (per hour per square foot). See the definition of the average value of a function on page 844.

22. We first find the area of the face of the surface that intersects the positive y-axis. A parametric representation of the surface is $x = x$, $y = \sqrt{1-z^2}$, $z = z$ with $x^2+z^2\le 1$. Then $\mathbf{r}(x,z) = \left\langle x, \sqrt{1-z^2}, z\right\rangle \Rightarrow \mathbf{r}_x = \langle 1,0,0\rangle$, $\mathbf{r}_z = \left\langle 0, -z/\sqrt{1-z^2}, 1\right\rangle$ and $\mathbf{r}_x\times\mathbf{r}_z = \left\langle 0,-1,-z/\sqrt{1-z^2}\right\rangle \Rightarrow |\mathbf{r}_x\times\mathbf{r}_z| = \sqrt{1+\frac{z^2}{1-z^2}} = \frac{1}{\sqrt{1-z^2}}$.

$$A(S) = \iint_{x^2+z^2\le 1}|\mathbf{r}_x\times\mathbf{r}_z|\,dA = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\frac{1}{\sqrt{1-z^2}}\,dx\,dz$$

$$= 4\int_0^1\int_0^{\sqrt{1-z^2}}\frac{1}{\sqrt{1-z^2}}\,dx\,dz \quad \text{(by the symmetry of the surface)}$$

8. $\mathbf{r}_u = \langle\cos v, \sin v, 0\rangle$, $\mathbf{r}_v = \langle -u\sin v, u\cos v, 1\rangle$, and $\mathbf{r}_u\times\mathbf{r}_v = \langle\sin v, -\cos v, u\rangle$. Then

$$A(S) = \int_0^{\pi}\int_0^1\sqrt{1+u^2}\,du\,dv = \int_0^{\pi}dv\int_0^1\sqrt{1+u^2}\,du$$

$$= \pi\left[\tfrac{u}{2}\sqrt{u^2+1} + \tfrac{1}{2}\ln\left|u+\sqrt{u^2+1}\right|\right]_0^1 = \tfrac{\pi}{2}\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$$

This integral is improper (when $z = 1$), so

$$A(S) = \lim_{t\to 1^-} 4\int_0^t \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\,dx\,dz = \lim_{t\to 1^-} 4\int_0^t \frac{\sqrt{1-z^2}}{\sqrt{1-z^2}}\,dz$$
$$= \lim_{t\to 1^-} 4\int_0^t dz = \lim_{t\to 1^-} 4t = 4$$

Since the complete surface consists of four congruent faces, the total surface area is $4\,(4) = 16$.

Alternate Solution: The face of the surface that intersects the positive y-axis can also be parametrized as $\mathbf{r}(x,\theta) = \langle x, \cos\theta, \sin\theta\rangle$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $x^2 + z^2 \le 1 \quad\Leftrightarrow\quad x^2 + \sin^2\theta \le 1 \quad\Leftrightarrow$ $-\sqrt{1-\sin^2\theta} \le x \le \sqrt{1-\sin^2\theta} \quad\Leftrightarrow\quad -\cos\theta \le x \le \cos\theta$. Then $\mathbf{r}_x = \langle 1,0,0\rangle$, $\mathbf{r}_\theta = \langle 0, -\sin\theta, \cos\theta\rangle$ and $\mathbf{r}_x \times \mathbf{r}_\theta = \langle 0, -\cos\theta, -\sin\theta\rangle \quad\Rightarrow\quad |\mathbf{r}_x \times \mathbf{r}_\theta| = 1$, so $A(S) = \int_{-\pi/2}^{\pi/2}\int_{-\cos\theta}^{\cos\theta} 1\,dx\,d\theta = \int_{-\pi/2}^{\pi/2} 2\cos\theta\,d\theta = 2\sin\theta\Big]_{-\pi/2}^{\pi/2} = 4$. Again, the area of the complete surface is $4(4) = 16$.