Problem 1) TF questions (80 points) Circle the correct letter. No justifications are needed.

T	F	A function $f(x,y)$ on the plane for which the absolute minimum and the absolute maximum are the same must be constant.

True. Remark. This would not be true if "absolute" would be replaced by "local".

T	F	The functions $f(x,y)$ and $g(x,y) = f(x,y) + 2002$ do not have the same critical points.

False. Because the gradients of f and g agree, also their critical points agree.

T	F	The sign of the Lagrange multiplier tells whether the critical point of $f(x,y)$ constrained to $g(x,y) = 0$ is a local maximum or a local minimum.

False. We would get the same Lagrange equations when replacing g with $-g$ and λ with $-\lambda$.

T	F	The gradient of a function $f(x,y,z)$ is tangent to the level surfaces of f

False. The gradient is normal to the level surface.

T	F	The point $(0,1)$ is a local minimum of the function $x^3+(\sin(y-1))^2$.

False. While the gradient is $(3x^2, 2\sin(y-1)\cos(y-1))$, the critical point is not a minimum.

T	F	For any curve, the acceleration vector $r''(t)$ of $r(t)$ is orthogonal to the velocity vector at $r(t)$.

False. Take $r(t) = (t^2, t)$. The velocity is $(2t, 1)$, the acceleration $(2, 0)$. Their dot product is $4t$.

T	F	If $D_u f(x,y,z) = 0$ for all unit vectors u, then (x,y,z) is a critical point.

True. If (x,y,z) is not a critical point, then the gradient vector $n = \nabla f(x,y,z)$ would have positive length and taking $u = n/||n||$ would give $D_u f(x,y,z) = ||n||^2 \neq 0$.

T	F	$\int_a^b \int_c^d x \, dxdy = (d^2 - c^2)(b-a)/2$, where a, b, c, d are constants.

True. Yes, by direct integration.

T	F	The functions $f(x,y)$ and $g(x,y) = (f(x,y))^2$ have the same critical points.

False. The gradient of g is $2f\nabla f$. So, the second function has critical points, where f vanishes.

T	F	If a function $f(x,y) = ax + by$ has a critical point, then $f(x,y) = 0$ for all (x,y).

True. At a critial point the gradient is $(a, b) = (0, 0)$, which implies $f = 0$.

T	F	$f_{xyxyx} = f_{yyxxx}$ for $f(x, y) = \sin(\cos(y + x^{14}) + \cos(x))$.

True. Follows from Clairot's theorem.

T	F	The function $f(x, y) = -x^{2002} - y^{2002}$ has a critical point at $(0, 0)$ which is a local minimum.

False. It is a local maximum.

T	F	It is possible that for some unit vector u, the directional derivative $D_u f(x, y)$ is zero even though the gradient $\nabla f(x, y)$ is nonzero.

True. This happens at a saddle point.

T	F	If (x_0, y_0) is the maximum of $f(x, y)$ on the disc $x^2 + y^2 \leq 1$ then $x_0^2 + y_0^2 < 1$.

False. The maximum could be on the boundary.

T	F	The linear approximation $L(x, y, z)$ of the function $f(x, y, z) = 3x + 5y - 7z$ at $(0, 0, 0)$ satisfies $L(x, y, z) = f(x, y, z)$.

True. $f(0, 0, 0) = 0$ and $\nabla f(0, 0, 0) = (3, 5, -7)$.

T	F	If $f(x, y) = \sin(x) + \sin(y)$, then $-\sqrt{2} \leq D_u f(x, y) \leq \sqrt{2}$.

True. $|D_u f| \leq ||\nabla f|| \leq \sqrt{2}$.

T	F	There are no functions $f(x, y)$ for which every point on the unit circle is a critical point.

False. There are many rotationally symmetric functions with this property.

T	F	An absolute maximum (x_0, y_0) of $f(x, y)$ is also an absolute maximum of $f(x, y)$ constrained to a curve $g(x, y) = c$ that goes through the point (x_0, y_0).

True. The Lagrange multiplier vanishes in this case.

T	F	If $f(x, y)$ has two local maxima on the plane, then f must have a local minimum on the plane.

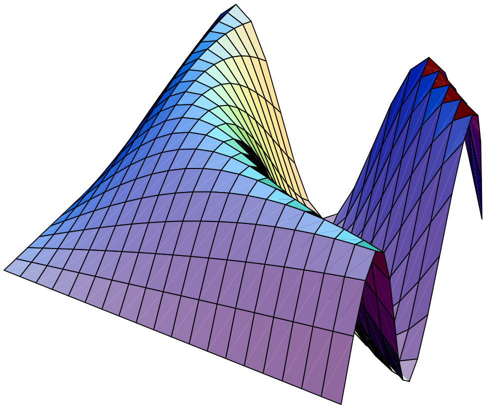
False. Look at a camel type surface. It has a saddle between the local maxima.

T	F	$\int\int_D f(x, y) g(x, y) \; dA = (\int\int_D f(x, y) \; dA)(\int\int_D g(x, y) \; dA)$ is true for all functions f and g.

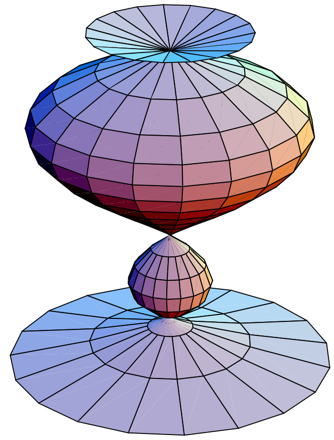
False. Example $f(x, y) = x^2, g(x, y) = x^3$ and where D is the unit square.

Problem 2) (30 points)

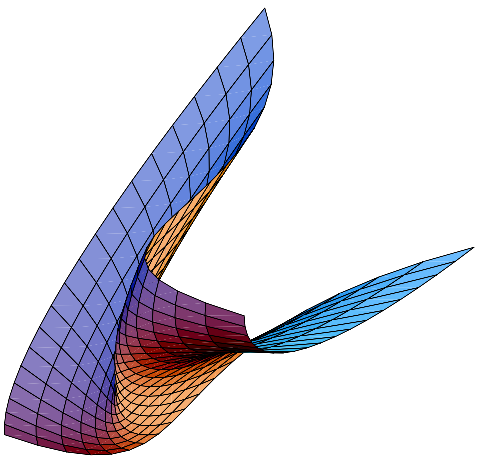
Match the parametric surfaces with their parameterization. No justification is needed.



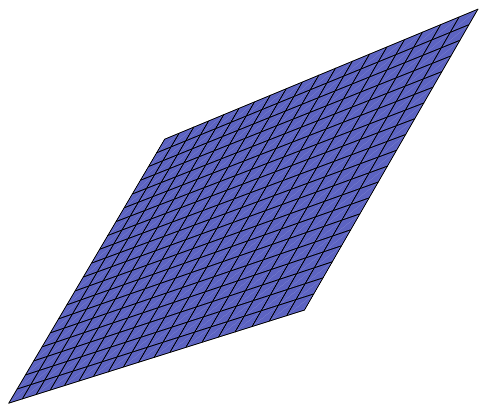
I



II



III



IV

Enter I,II,III,IV here	Parameterization
IV	$(u,v) \mapsto (u,v,u+v)$
I	$(u,v) \mapsto (u,v,\sin(uv))$
II	$(u,v) \mapsto (0.2+u(1-u^2))\cos(v),(0.2+u(1-u^2))\sin(v),u)$
III	$(u,v) \mapsto (u^3,(u-v)^2,v)$

Surface I is a graph.
Surface II is a surface of revolution.
Surface III is algebraic. One of the traces is (u^3,u^2), an other trace is the parabola (v^2,v).
Surface IV is a plane.

Problem 3) (40 points)

Match the integrals with those obtained by changing the order of integration. No justifications are needed.

Enter I,II,III,IV or V here.	Integral
V	$\int_0^1 \int_{1-y}^1 f(x,y)\ dxdy$
I	$\int_0^1 \int_y^1 f(x,y)\ dxdy$
II	$\int_0^1 \int_0^{1-y} f(x,y)\ dxdy$
III	$\int_0^1 \int_0^y f(x,y)\ dxdy$

I) $\int_0^1 \int_0^x f(x,y)\ dydx$

II) $\int_0^1 \int_0^{1-x} f(x,y)\ dydx$

III) $\int_0^1 \int_x^1 f(x,y)\ dydx$

IV) $\int_0^1 \int_0^{x-1} f(x,y)\ dydx$

V) $\int_0^1 \int_{1-x}^1 f(x,y)\ dydx$

Problem 4) (40 points)

Consider the graph of the function $h(x,y) = e^{-3x-y} + 4$.

1. Find a function $g(x,y,z)$ of three variables such that this surface is the level set of g.
2. Find a vector normal to the tangent plane of this surface at (x,y,z).
3. Is this tangent plane ever horizontal? Why or why not?
4. Give an equation for the tangent plane at $(0,0)$.

Solution.

1. $g(x, y, z) = e^{-3x-y} + 4 - z$.
2. $\nabla g(x, y, z) = (3e^{3x_0-y_0}, -e^{3x_0-y_0}, -1)$. At the point (x_0, y_0, z_0), we have the gradient $(a, b, c) = (-3e^{-3x_0-y_0}, -e^{3x_0-y_0}, -1)$ and so the plane $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$.
3. Horizontal would mean $a = b = 0$ which is not possible because $-e^{3x_0-y_0}$ is always negative.
4. The tangent plane which goes through the point $(0, 0, h(0, 0)) = (0, 0, 5) = (x_0, y_0, z_0)$ is $-3x - y - z = d$, where $d = 30 - 10 - 15 = -5$. $\boxed{3x + y + z = 5}$.

Problem 5) (40 points)

Find all the critical points of the function $f(x, y) = \frac{x^2}{2} + \frac{3y^2}{2} - xy^3$. For each, specify if it is a local maximum, a local minimum or a saddle point and briefly show how you know.

Solution. $\nabla f(x, y) = \langle x - y^3, 3y - 3xy^2\rangle$. This is zero if $3y - 3y^5 = 0$ or $y(1 - y^4) = 0$ which means $y = 0$ or $y = \pm 1$. In the case $y = 0$, we have $x = 0$. In the case $y = 1$, we have $x = 1$, in the case $y = -1$, we have $x = -1$. The critical points are $(0, 0), (1, 1), (-1, -1)$.

The discriminant is $f_{xx}f_{yy} - f_{xy}^2 = 3 - 9y^4$. The entry f_{xx} is 1 everywhere.

Applying the second derivative test gives

Critical point	(0,0)	(1,1)	(-1,-1)
Discriminant	3	-6	-6
f_{xx}	1	1	1
Anslysis	min	saddle	saddle

Problem 6) (40 points)

Minimize the function $E(x, y, z) = \frac{k^2}{8m}(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2})$ under the constraint $xyz = 8$, where k^2 and m are constants.

Remark. In quantum mechanics, E is the ground state energy of a particle in a box with dimensions x, y, z. The constant k is usually denoted by $\hbar$ and called the Planck constant.

Solution. Write $C = k^2/(8m)$ to save typing. $\nabla E(x, y, z) = -2C(1/x^3, 1/y^3, 1/z^3)$. The constraint is $G(x, y, z) = xyz - 8 = 0$. We have $\nabla G(x, y, z) = (yz, xz, xy)$. The Lagrange equations are

$$2C = \lambda x^3 yz$$

$$\begin{aligned} 2C &= \lambda x y^3 z \\ 2C &= \lambda x y z^3 \\ xyz &= 8 \end{aligned}$$

Eliminating λ gives $x^2 = y^2 = z^2$ and $x = y = z = 2$ and the minimal energy is $\boxed{3C/4 = 3k^3/(32m)}$.

Problem 7) (40 points)

Assume $F(x,y) = g(x^2 + y^2)$, where g is a function of one variable. Find $F_{xx}(1,2) + F_{yy}(1,2)$, given that $g'(5) = 3$ and $g''(5) = 7$.

Solution.
$F_x = g'(x^2 + y^2)2x$.
$F_{xx}(x,y) = g''(x^2 + y^2)4x^2 + g'(x^2 + y^2)2$.
$F_y = g'(x^2 + y^2)2y$.
$F_{yy}(x,y) = g''(x^2 + y^2)4y^2 + g'(x^2 + y^2)2$.

$F_{xx} + F_{yy}(1,2) = 7 \cdot 4 \cdot 5 + 3(2+2) = \boxed{152}$.

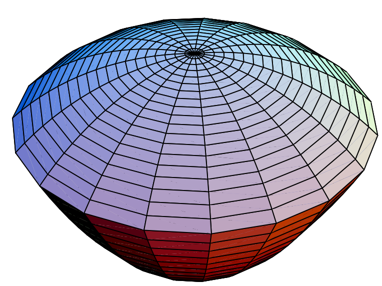
Problem 8) (40 points)

Consider the region inside $x^2 + y^2 + z^2 = 2$ above the surface $z = x^2 + y^2$.

a) Sketch the region.

b) Find its volume.

Solution.



a) The intersection of the two surfaces is a circle of radius 1. The region is the bottom of a paraboloid covered with a spherical cap.

b) Use polar coordinates: $2\pi \int_0^1 (\sqrt{2 - r^2} - r^2) r\,dr = -(\pi/3)(2 - r^2)^{3/2}|_0^1 - \pi/2 = \boxed{(\pi/3)(2^{3/2} - 1) - \pi/4}$.