Problem 1) TF questions (80 points)

| T | F |

At a local maximum (x_0, y_0) of $f(x, y)$, one has $f_{yy}(x_0, y_0) \geq 0$.

False. At a local maximum, $f_{yy} \leq 0$.

| T | F |

If R is the region bounded by $x^2 + 4y^2 = 1$ then $\int \int_R xy^4 \, dxdy < 0$.

False. The integral is zero because the integral on $R \cap \{x > 0\}$ is the negative of $R \cap \{x < 0\}$.

| T | F |

The gradient $\langle 2x, 2y \rangle$ is perpendicular to the surface $z = x^2 + y^2$.

False. The surface is the graph of a function $f(x, y)$. While the gradient of f is perpendicular to the level curve of f, it is only the projection of the gradient to the function $g(x, y, z) = f(x, y) - z$. The later is perpendicular to the surface.

| T | F |

The equation $f(x, y) = k$ implicitly defines x as a function of y and $\frac{dx}{dy} = \frac{\partial f}{\partial y} / \frac{\partial f}{\partial x}$.

False. Almost right, the sign is wrong.

| T | F |

$f(x, y) = \sqrt{(16 - x^2 - y^2)}$ has both an absolute maximum and an absolute minimum on its domain of definition.

True. The domain of definition is the disc $x^2 + y^2 \leq 16$. The maximum 4 is in the center the absolute minimum 0 at the boundary.

| T | F |

If (x_0, y_0) is a critical point of $f(x, y)$ under the constraint $g(x, y) = 0$, and $f_{xy}(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point.

False. The point (x_0, y_0) does not need to be a critical point of f at all.

| T | F |

The vector $r_u(u, v)$ of a parameterized surface $(u, v) \mapsto r(u, v) = (x(u, v), y(u, v), z(u, v))$ is normal to the surface.

False. The vector is always tangent to the surface.

| T | F |

The identity $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) \, dydx = \int_0^1 \int_0^{\pi/2} r^2 \, d\theta dr$ holds.

False. The area element $d\theta dr$ should be replaced by $rd\theta dr$. So, the right hand side should be $\int_0^1 \int_0^{\pi/2} r^3 \, d\theta dr$.

| T | F |

$f(x, y)$ and $g(x, y) = f(x^2, y^2)$ have the same critical points.

False. The function g has always $(0, 0)$ as a critical point, even if f has not.

| T | F |

If $f(x, t)$ satisfies the Laplace equation $f_{xx} + f_{tt} = 0$ and simultaneously the wave equation $f_{xx} = f_{tt}$, then $f(x, t) = ax + bt + c$.

False. Take $f(x, t) = xt$. (Here is how we get the general solution: From the two equations, we get $f_{xx} = 0$ and $f_{tt} = 0$. From $f_{xx} = 0$, we obtain that $f(x, t) = a(t)x + c(t)$. From $f_{tt} = 0$, we obtain $a(t)$ and $c(t)$ are linear in t. Therefore the general solution is $f(x, t) = atx + bt + cx + e$).

| T | F |

Every smooth function satisfies the partial differential equation $f_{xxyy} = f_{xyxy}$.

True. This follows from Clairot's theorem.

| T | F |

The function $f(x, y) = (x^4 - y^4)$ has neither a local maximum nor a local minimum at $(0, 0)$.

True. The function is both smaller and bigger than $f(0, 0)$ for points near $(0, 0)$.

| T | F |

$\int_0^1 \int_0^{\pi/2} r \, d\theta dr = \pi/4$.

True. This is the area of a quarter of the unit disc.

| T | F | At a saddle point, the directional derivative is zero for two different vectors u, v.

True. The directional derivative can be both positive and negative at a saddle point. By the intermediate value theorem, there are two directions, where the directional derivative vanishes.

| T | F | It is possible to find a function of two variables which has no maximum and no minimum.

True. There are many linear functions like that.

| T | F | The value of the function $f(x, y) = e^x y$ at $(0.001, -0.001)$ can by linear approximation be estimated as -0.001.

True. Because the gradient at $(0, 0)$ is $(0, 1)$ and $f(0, 0) = 0$, the linear approximation is $L(x, y) = y$.

| T | F | For any function $f(x, y, z)$ and any unit vectors u, v, one has the identity $D_{u \times v} f(x, y, z) = D_u f(x, y, z) D_v f(x, y, z)$.

False. The directional derivative in the $u \times v$ direction has nothing to do with the directional derivatives into the other directions. An example, $u = (1, 0, 0), v = (0, 1, 0), f(x, y, z) = x + y$ is an example, where $D_{u \times v} f(x, y, z) = 0$ but $D_u f = 1, D_v f = 1$.

| T | F | Given 2 arbitrary points in the plane, there is a function $f(x, y)$ which has these points as critical points and no other critical points.

True If $(a, b), (c, d)$ are the two points, we want $\nabla f(x, y) = \langle (x - a)(x - c), (y - b)(y - d) \rangle$. So, take $f(x, y) = x^3/3 - (a + c)x^2 + acx + y^3/3 - (b + d)y^2 + bdy$.

| T | F | The maximum of $f(x, y)$ under the constraint $g(x, y) = 0$ is the same as the maximum of $g(x, y)$ under the constraint $f(x, y) = 0$.

False This can not be true, because the first problem is the same if we replace $g(x, y)$ with $2g(x, y)$, but this will change the value of the maximum of g on the right hand side.

| T | F | Assume (x_0, y_0) is a critical point of $f(x, y)$ and $f_{xx} f_{yy} - f_{xy}^2 \neq 0$ at this point. Let T be the tangent plane of the surface $S = \{f(x, y) - z = 0\}$ at $P = (x_0, y_0, f(x_0, y_0))$. If the intersection of T with S is a single point, then (x_0, y_0) is a local max or local min.
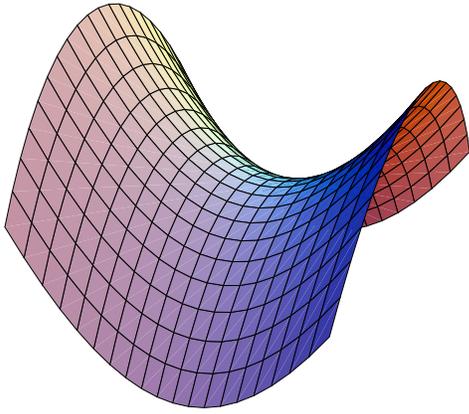
True. The other possibility would be a saddle point, in which case, the tangent space intersects the surface in two curves which pass through the critical point.
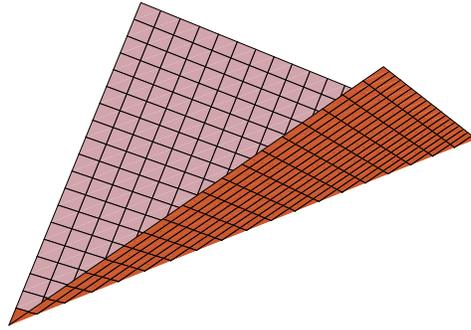
The Key is

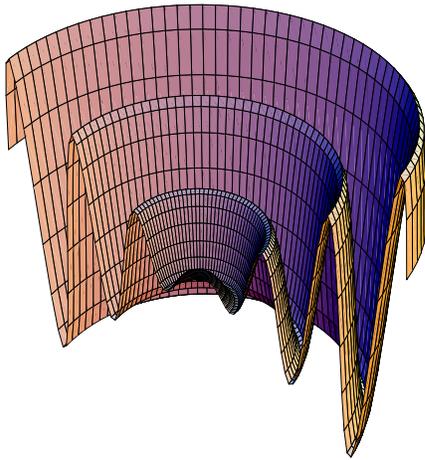| F F F F T F F F F F T T T T T T T F T F T |

Problem 2) (30 points)

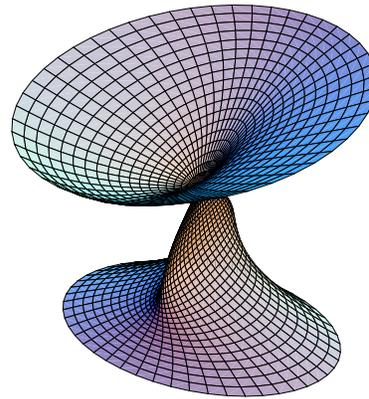Match the parametric surfaces with their parameterization. No justification is needed.

I

II

III

IV

Enter I,II,III,IV here	Parameterization
III	$(u,v) \mapsto (u\cos(v), u\sin(v), u^2\cos(u)/(u+1))$
II	$(u,v) \mapsto (u, v+u, |u-v|)$
IV	$(u,v) \mapsto ((u-\sin(u))\cos(v), (u-\cos(u))\sin(v), u)$
I	$(u,v) \mapsto (u, v, u^2 - v^2)$

Find all the critical points of the function $f(x, y) = xy(4 - x^2 - y^2)$. Are they maxima, minima or saddle points?

Solution. Taking derivatives of $f(x, y) = 4xy - x^3y - xy^3$ gives $\nabla f(x, y) = (4y - 3x^2y - y^3, 4x - x^3 - 3xy^2)$. To solve the system

$$y(4 - 3x^2 - y^2) = 0 \tag{1}$$
$$x(4 - x^2 - 3y^2) = 0 \tag{2}$$

We have the four following possibilites:

1) $y = 0, x = 0$
2) $4 - 3x^2 - y^2 = 0, x = 0$
3) $4 - x^2 - 3y^2 = 0, y = 0$
4) $4 - 3x^2 - y^2 = 0, 4 - x^2 - 3y^2 = 0$.

There are 9 critical points in total

1) gives the critical point $(0, 0)$.
2) gives the critical points $(0, 2), (0, -2)$.
3) gives the critical points $(2, 0), (-2, 0)$.
4) (subtract 3 times the second equation from the first): $(1, 1), (-1, 1), (1, 1), (1, -1)$.

The Hessian determimant (=discriminant) $f_{xx}f_{yy} - f_{xy}^2$ at a general point is $-9(x^4 + y^4) - 16 + 24(x^2 + y^2) + 18x^2y^2$ and $f_{xx}(x, y) = -6xy$.

Applying the second derivative test gives

Critical point	(-2, 0)	(-1, -1)	(-1, 1)	(0, 0)	(1, -1)	(1, 1)	(2, 0)	(-2, 0)	(2, 0)
Discriminant	-64	32	32	-16	32	32	-64	-64	-64
f_{xx}	0	-6	6	0	6	-6	0	0	0
Anslysis	saddle	max	min	saddle	min	max	saddle	saddle	saddle

Let $f(x, y) = e^{(x-y)}$ so that $f(\log(2), \log(2)) = 1$. Find the equation for the tangent plane to the graph of f at $(\log(2), \log(2))$ and use it to estimate $f(\log(2) + 0.1, \log(2) + 0.004)$.

Solution. The graph of f is a level curve of the function $g(x, y, z) = z - f(x, y)$. The gradient at the point $(x_0, y_0, f(x_0, y_0)) = (\log(2), \log(2), 1)$ is $(a, b, c) = (-1, 1, 1)$, so that the tangent plane has an equation $ax + by + cz = -x + y + z = d$. and the constant d is obtained from $d = -x_0 + y_0 + z_0 = 1$. Therefore

$$-x + y + z - 1 = 0$$

At the point $(\log(2), \log(2), 1)$, the level surface $g = 0$ is close to the level surface $L(x, y, z) = x - y - z + 1 = 0$. If we plug in $x = 0.1, y = 0.04, z = 0$, we get 1.06.

Remark. We could have stayed in two dimensions and estimate $f(x_0 + dx, f(y_0 + dx)$ by $f(x_0, y_0) + (1, -1) \cdot (dx, dy) = 1 + dx - dy$ which is for $dx = 0.1, dy = 0.04$ equal to $1 + 0.1 - 0.04 = \boxed{1.06}$.

Problem 5) (40 points)

f is a function which depends on x and y, where $x = u^3 v$ and $y = u^2 v^2$. When $(u, v) = (1, 1)$ $\frac{\partial f}{\partial x} = -5$, $\frac{\partial f}{\partial v} = 9$. What is $\frac{\partial f}{\partial u}$?

Solution. Apply the chain rule to $(u, v) \mapsto f(x(u, v), y(u, v)) = f(u^3 v, u^2 v^2)$:

$$f_u(u, v) = f_x(x(u, v), y(u, v))x_u(u, v) + f_y(x(u, v), y(u, v))y_u(u, v)$$
$$f_v(u, v) = f_x(x(u, v), y(u, v))x_v(u, v) + f_y(x(u, v), y(u, v))y_v(u, v)$$

Using $f_x(1, 1) = -5, f_v(1, 1) = 9, x_u(u, v) = 3, x_v(1, 1) = 1, y_u(1, 1) = 2, y_v(1, 1) = 2$ these two equations are

$$f_u(1, 1) = (-5)3 + f_y(1, 1)2$$
$$9 = (-5)1 + f_y(1, 1)2$$

The second equation can be solved for $f_y(1, 1) = 7$. Plugging this into the first equation gives $f_u(1, 1) = \boxed{-1}$.

Problem 6) (40 points)

A **can** is a cylinder with a circular base. Its surface area (top, bottom and sides) is 300π cm^2. What is the maximum possible volume of such a can?

Solution. We have the problem to extremize $f(r, h) = \pi r^2 h$ under the constraint $2\pi r^2 + 2\pi rh = 300\pi$. This is equivalent to extremize $f(r, h) = \pi r^2 h$ under the constraint $g(r, h) = r^2 + rh = 150$.

The Lagrange equations are

$$
\begin{aligned}
2\pi rh &= \lambda(2r + h) \\
\pi r^2 &= \lambda r \\
r^2 + rh &= 150
\end{aligned}
$$

The second equation gives $\pi r = \lambda$. Plugging in λ into the first equation gtives $h = 2r$. From the last equation, we get $r^2 + 2r^2 = 150$ or $r^2 = 50$. Therefore $r = 5\sqrt{2}cm, h = 10\sqrt{2}cm$. The maximal volume is $\pi r^2 h = \boxed{\pi 500\sqrt{2}}$.

Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{xy^5}{x^2+y^2} \, dy \, dx$.

Solution. The integral is taken over the disc intersected with the first quadrant in the plane. In Polar coordiantes $x^2 + y^2 = r, x = r\cos(\theta), y = r\sin(\theta)$ the integral is therefore

$$\int_0^2 \int_0^{\pi/2} \frac{r\cos(\theta) r^5 \sin^5(\theta)}{r^2} \, rd\theta dr = \int_0^2 \int_0^{\pi/2} r^5 \cos(\theta) \sin^5(\theta) \, d\theta dr = 64/36 = 16/9$$

Solution $\boxed{16/9}$.

Problem 8) (40 points)

a) Find the area of the region D enclosed by the lines $x = \pm 2$ and the parabolas $y = 1 + x^2$, $y = -1 - x^2$.

b) Find the integral of $f(x, y) = y^2$ on the same region as in a). (The result can be interpreted as a moment of inertia).
Solution.

a) $\int_{-2}^2 \int_{-1-x^2}^{1+x^2} 1 \, dydx = \int_{-2}^2 2 + 2x^2 \, dx = 8 + 2x^3/3|_{-2}^2 = 8 + 32/3 = \boxed{56/3}$.
b) $\int_{-2}^2 \int_{-1-x^2}^{1+x^2} y^2 \, dydx = \int_{-2}^2 2(1+x^2)^3/3 \, dx = \int_{-2}^2 (2 + 6x^2 + 6x^4 + 2x^6)/3 \, dx = \boxed{2216/35}$