An Excursion into Algebraic Geometry.

A **rational** function is any function $f(t) = \frac{p(t)}{q(t)}$ where $p(t)$ and $q(t)$ are polynomials having no common zeroes.

Theorem. Let $S \subset \mathbb{R}^2$ denote the hyperbola $y^2 - x^2 + 1 = 0$. Then, there exists a parameterization of S whose parametric equations are rational functions.
Proof. Let L_t denote the line through the point $(1, 0)$ on S having slope S. Let us parameterize the line L_t as follows: $L_t = (1, 0) + \lambda(1, t) = (\lambda + 1, t\lambda)$ where $\lambda \in \mathbb{R}$. It is easy to see that L_t will intersect S at exactly one other point. We need to find this other point of intersection. To do so, we substitute the parameterization into the equation for the hyperbola and solve for λ:

$$\begin{aligned} (t\lambda)^2 - (\lambda + 1)^2 + 1 &= 0 \\ t^2\lambda^2 - \lambda^2 - 2\lambda &= 0 \\ (t^2 - 1)\lambda^2 - 2\lambda &= 0, \end{aligned}$$

which means that $\lambda = 0$ and $\lambda = \frac{2}{t^2-1}$ are the roots. Notice that $\lambda = 0$ corresponds to the point $(1, 0)$ on the hyperbola. Thus, the second point of intersection of L_t and S is found by plugging in $\lambda = \frac{2}{t^2-1}$ into the parameterization for the line L_t:

$$\left(\frac{t^2 + 1}{t^2 - 1}, \frac{2t}{t^2 - 1} \right).$$

Note that $x(t) = \frac{t^2+1}{t^2-1}$ and $y(t) = \frac{2t}{t^2-1}$ are both rational functions, as desired. Formally, we have to allow the point $t = \infty$ to recover $(1, 0)$. Nevertheless, the given parametric equations will hit all points on S. Moreover, $y(t)^2 - x(t)^2 + 1$ vanishes identically—confirming the validity of our parameterization. ∎

Exercise. Let $S \subset \mathbb{R}^2$ denote the circle $x^2 + y^2 - 1 = 0$. Find a rational parameterization of S.