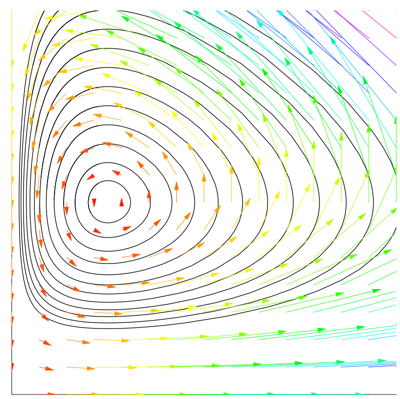
CURVES, TANGENTS AND DIFFERENTIAL EQUATIONS Math21a, Fall 2002

INTRODUCTION. So far, we have seen how, given a smooth parameterized curve, you can compute the tangent vector to that curve at each point. In many applications, the inverse problem is the essential question. Namely, we are given a vector at each point and want to find curves that have these vectors as their tangents. The picture of all these vectors $\langle f(x,y), g(x,y)\rangle$ with their tails at (x,y) is called a **vector field**. Sketching curves $\langle x(t), y(t)\rangle$ whose tangents belong to a vector field like this is actually a powerful approach to studying differential equations since the prescribed tangent vectors, say $\langle f(x,y), g(x,y)\rangle$ at (x,y) are telling us, what $\langle dx/dt, dy/dt\rangle$ have to be. You may have thought before that the only way to deal with differential equations was to write out formulas for particular solutions given initial conditions. The examples that follow illustrate how useful it can be to picture the solutions - all of them at once - as a family of curves.

EXAMPLE 1

Let $x(t)$ denote the population of a "prey species" like for example tuna fish. Its growth $x'(t) = ax(t)+bx(t)y(t)$ with positive parameters a,b has a contribution proportional to x and a contribution proportional to xy, where $y(t)$ is the population size of a "predator" like for example sharks (who like to eat tuna fish). The reason for the second term is that more predators and more prey species will lead to prey consumption. The growth of the predator population $y(t)$ is assumed to be proportional to $-cy(t)+dxy$, where c,d are positive. The reason for a negative sign of the first term is that the predator would die out without food. The reason for the second term is that both more predators as well as more prey leads to a growth of predators through reproduction. A concrete example is

$$\begin{aligned} \dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy \end{aligned}$$

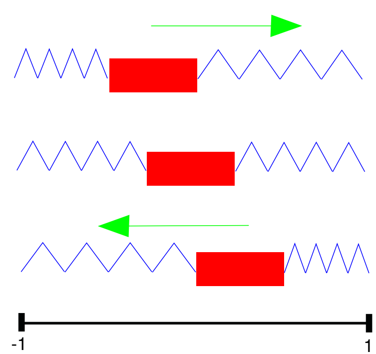


It is an example of a **Volterra-Lodka system**. Volterra explained with such systems the first time the oscillation of fish populations in the Mediterranean sea. At any specific point $(x,y) = (x(t), y(t))$, there is a unique curve $r(t) = (x(t), y(t))$ through that point for which the tangent $r'(t) = (x'(t), y'(t)$ is the vector
$(0.4x - 0.4xy, -0.1y + 0.2xy)$.

EXAMPLE 2

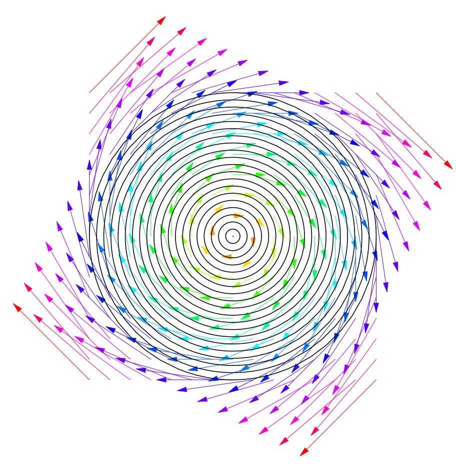
Newton's law $m\vec{r}'' = F$ relates the acceleration $\vec{r}''$ of a body at a specific point with the force F acting at the point. For example, if $x(t)$ is the position of a mass point in $[-1,1]$ attached at two springs and the mass is $m=2$, then the point experiences a force $(-x+(-x)) = -2x$ so that $mx'' = 2x$ or

$$x''(t) = -x(t) \ .$$



If we introduce a new function $y(t) = x'(t)$ of t, then $x'(t) = y(t)$ and $y'(t) = -x(t)$. Of course y is the velocity of the mass point, so a pair (x,y), thought of as an initial condition, describes the system so that nature knows what the future evolution of the system has to be given that data.

We don't yet know the curve $t \mapsto (x(t), y(t))$, but we know the tangents $(x'(t), y'(t)) = (y(t), -x(t))$. In other words, we know a direction at each point. The equation $(x' = y, y' = -x)$ is called a system of ordinary differential equations (ODE). More generally, the problem when studying ODE's is to find solutions $x(t), y(t)$ of equations $x'(t) = f(x(t), y(t)), y'(t) = g(x(t), y(t))$. Here we look for curves $x(t), y(t)$ so that at any given point (x,y), the tangent vector $(x'(t), y'(t))$ is $(y, -x)$. You can check by differentiation that the circles $(x(t), y(t)) = (r\sin(t), r\cos(t))$ are solutions. They form a family of curves. Can you interpret these solutions physically?



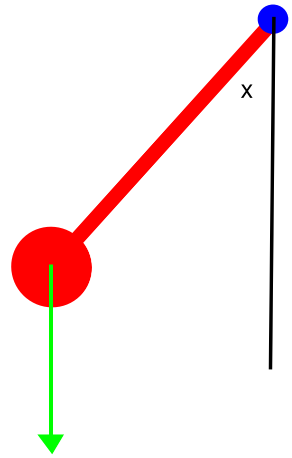
EXAMPLE 3

If $x(t)$ is the angle of a pendulum, then the gravity acting on it produces a force

$$F(x) = -gm\sin(x) ,$$

where m is the mass of the pendulum and where g is a constant. For example, if $x = 0$ (pendulum at bottom) or $x = \pi$ (pendulum at the top), then the force is zero.
The Newton equation "mass times acceleration = Force" gives

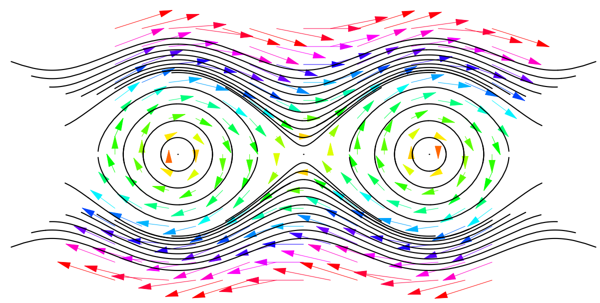
$$\ddot{x}(t) = -g\sin(x(t)) .$$



The equation of motion for the pendulum $\ddot{x}(t) = -g\sin(x(t))$ can be written with $y = \dot{x}$ also as

$$\frac{d}{dt}(x(t), y(t)) = (y(t), -g\sin(x(t))) .$$

Each possible motion of the pendulum $x(t)$ is described by a curve $r(t) = (x(t), y(t))$. Writing down explicit formulas for $(x(t), y(t))$ is in this case not possible with known functions like $\sin, \cos, \exp, \log$ etc. However, one still can understand the curves:



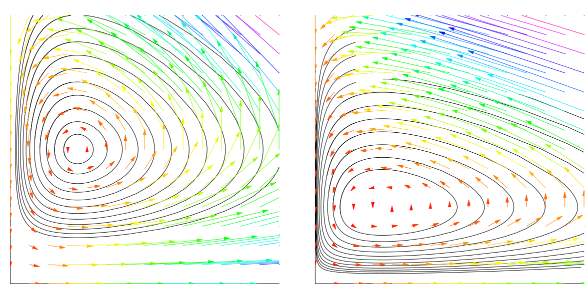
Curves on the top of the picture represent situations where the velocity y is large. They describe the pendulum spinning around fast in the clockwise direction. Curves starting near the point $(0, 0)$, where the pendulum is at a stable rest, describe small oscillations of the pendulum. The curves look there similar as the curves in Example 2).

PROBLEMS.

Problem 1) The population model

$$\begin{aligned} \dot{x} &= 0.4x - bxy \\ \dot{y} &= -0.1y + 0.2xy \end{aligned}$$

is shown for $b = 0.4$ and $b = 0.7$. Explain the change in the phase portraits.



Problem 2) Sketch the curve $t \mapsto r(t) = (x(t), y(t)) = (e^{-t}\cos(t), e^{-t}\sin(t))$ in the plane. Find differential equations $(x'(t), y'(t)) = (f(x, y), g(x, y))$ which contains the curve $r(t)$ as a solution. Find an equivalent second order differential equation $x''(t) = F(x(t), x'(t))$ and a physical interpretation of this system.

Problem 3) Consider in example 3 above the modification, where the pendulum has some friction proportional to the speed. Sketch how the solution curves and the vector field look like now in the $x - y$ plane. If you want to use equations, then consider $\ddot{x} = -g\sin(x) - k\dot{x}$ with $k > 0$.