

INTEGRATION AND DENSITIES

1. INTRODUCTION

In this course we study many different kinds of integrals. However, all of them share something in common: All integrals are computed to measure some quantity which is additive. Let us explain what this means. If we have a curve (which you should think of as a piece of wire bent around in space), then we can obtain the length of this curve by cutting it up into pieces, taking the length of each piece, and adding up all the lengths to get the total length. Similarly if we were measuring the area of a surface (which can be thought of as a sheet of rubber bent around in space), we could cut up our surface into pieces, somehow measure the area of each piece, and add up all the areas to get the total area.

Hence an additive quantity is a quantity whose value on the union of a set of pieces is equal to the sum of its value on each piece. Here are some more examples: the mass of an object is equal to the sum of the masses of the pieces that make it up. Also the electric charge on an object can be obtained by adding up the charge on its constituent parts. A less trivial example you may have seen is *probability*. If a collection of numbers (like course grades) is distributed according to some *probability distribution* (like the bell curve) then the probability that a number falls within two disjoint intervals (in one or the other) is equal to the sum of the probabilities that it falls within each of the intervals. There are many more examples, such as circulation of a fluid around a curve and the flux of a fluid through a surface. We will describe these in more detail later.

Again, the common theme is we are trying to measure something which can be measured on constituent parts individually and then summed. Not all physical quantities are additive. For example, if we took a one litre bucket of water at 30 degrees temperature and added it to a two litre bucket of water at 60 degrees temperature, then we would have three litres of water (volume *is* additive) but the temperature of the combined volume would certainly not be the sum of the two temperatures.

Sometimes we measure two different additive quantities and compare their ratios. For example, the average *linear mass density* of a curve is the total mass of the curve divided by its length. Similarly the average *surface mass density* of a surface is the total mass of the surface divided by its area, and the average mass density of a solid object is equal to its total mass divided by its volume. In the same way one can talk about charge densities or the density of “flux per unit volume” or “circulation per unit area.” Again, we will be more precise about these ideas later.

There are two fundamental principles we can use to understand the different kinds of integration and the fundamental theorems they correspond to.

- **The Locally Linear Principle.** On a *small scale*, curves are very close to straight lines and surfaces are very close to planes. Of course, this is not true for all curves and surfaces (for example those with sharp corners or tears or holes). It *is*, however, true for the “smooth” curves and surfaces

we are going to integrate over. If we chop up these curves and surfaces into very small pieces, we can reasonably well approximate those pieces with lines and planes, and our approximation gets better in the limit as the size (length or area) of these pieces gets smaller. It is important to stop and realize why this principle is so useful. We want to be able to calculate additive quantities such as mass and charge for large extended objects like curves and surfaces by adding up the quantity on small constituent pieces of our larger objects. This principle tells us how we can calculate the length or area for very small pieces of curves and surfaces, by *approximating* them by line segments and parts of planes. We know how to get the lengths of straight lines and the areas of planes.

- **The Density Principle.** In the limit as the size of our constituent piece goes to zero (and so the piece becomes a point), the average density of that piece approaches some limit which we call the density at that point. This gives us a *density function* since we have a density at each point. Then for most reasonable physical quantities, the density can be assumed to be continuous. That is, if we look at two very close points in our object, the densities at those points will be close. So in a small piece, the density is approximately constant, and the quantity we are measuring will be approximately equal to the size of the piece (length, area, or volume) multiplied by the density at any point in the piece. This principle can now be combined with the first one in a very powerful way. Since the Locally Linear Principle tells us how to calculate (approximately) the length of a small piece of a curve or the area of a small piece of a surface, combining this fact with the Density Principle tells us how to calculate the mass or charge or any other additive quantity of a curve or surface. For small pieces, we simply multiply the length or area by the density (per unit length or unit area) of the additive quantity in that small piece, which is approximately constant.

We can relate integration and densities in two ways: Let us take an object M (curve, surface, or solid body) with density function $\sigma(x, y, z)$. Suppose we are measuring some quantity Q_M . If we chop up M into lots of little pieces M_i of *small* size ΔS_i , $i = 1, \dots, N$, then the quantity Q_i of piece M_i is approximately $Q_i = \sigma(\mathbf{r}_i)\Delta S_i$ where \mathbf{r}_i is any point in M_i . Here “size” refers to length for curves, area for surfaces, and volume for three-dimensional solid regions. Then

$$Q_M = \sum_{i=1}^N Q_i \approx \sum_{i=1}^N \sigma(\mathbf{r}_i)\Delta S_i \rightarrow \int_M \sigma(x, y, z)dS \quad \text{as } N \rightarrow \infty$$

This tells us that to obtain the total quantity Q_M of the object M , we integrate the density function $\sigma(\mathbf{r})$ over the object. Conversely, if our object M has small size S_M , then $Q_M \approx \sigma(\mathbf{r})S_M$ for any point \mathbf{r} in M . So we can obtain the density function $\sigma(\mathbf{r})$ from

$$\sigma(\mathbf{r}) = \lim_{M_r \rightarrow \mathbf{r}} \frac{Q_{M_r}}{S_{M_r}}$$

where M_r is a small piece containing the point \mathbf{r} . We will illustrate these principles many times in the rest of this handout with specific examples.

2. INTEGRALS OVER CURVES

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a parametrized curve, for $a < t < b$. We can cut up the interval $[a, b]$ into N pieces P_1, \dots, P_N of length Δt_i , where N is taken to be very large and each Δt_i is small. Over a small range of the parameters, by the Locally Linear Principle, the curve can be approximated by a straight line in the direction of the vector $\mathbf{r}'(t_i)$ for any point t_i in P_i . Thus the *length* L_i of this piece is approximately $|\mathbf{r}'(t_i)|\Delta t_i$. Hence the total length of the curve is

$$L = \lim_{N \rightarrow \infty} \sum_{i=1}^N |\mathbf{r}'(t_i)|\Delta t_i = \int_a^b |\mathbf{r}'(t)| dt$$

If we are measuring some other quantity Q (like mass or charge) whose amount per unit length of the curve is the linear density function $\sigma(x, y, z)$, then the Density Principle tells us that the amount Q_i in the piece corresponding to the parameter range P_i is $\sigma(\mathbf{r}(t_i))|\mathbf{r}'(t_i)|\Delta t_i$, and hence the total quantity Q is:

$$Q = \int_a^b \sigma(\mathbf{r}(t))|\mathbf{r}'(t)| dt$$

Another kind of integral we can do over a curve which is also given a *prescribed orientation*, or direction of travel, is the integral of a *vector field* \mathbf{F} , also called a *line integral*. Suppose that \mathbf{F} is the velocity field of a fluid. Then at any point along a curve, the component of the fluid's velocity in the direction of the curve is given by $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{T}$ where \mathbf{T} is the unit tangent vector to the curve in the direction of our given orientation. We can multiply this by a little piece of length L_i to get a number proportional to the amount of fluid moving in the direction of the curve at that point:

$$(\mathbf{F}(\mathbf{r}(t_i)) \cdot \mathbf{T})L_i = \mathbf{F}(\mathbf{r}(t_i)) \cdot \frac{\mathbf{r}'(t_i)}{|\mathbf{r}'(t_i)|} |\mathbf{r}'(t_i)|\Delta t_i = \mathbf{F}(\mathbf{r}(t_i)) \cdot \mathbf{r}'(t_i)\Delta t_i$$

Adding up all the contributions over the whole curve, we get the *circulation* of the vector field \mathbf{F} around the curve:

$$\text{Circulation} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

This measures the tendency of the fluid to flow around in the direction of the curve. In Section ?? we will see how we can use the density principle to derive an expression for the circulation per unit area, a density function which we can integrate over surfaces. In the case when \mathbf{F} is a force field, the line integral of \mathbf{F} over a curve gives the work done in moving a particle through that path in the presence of that force field. In practice computing line integrals of vector fields is easier than integrals of functions over curves, because there is no square root term which arises from $|\mathbf{r}'(t)|$ in the function case.

3. INTEGRALS OVER SURFACES

Let $\mathbf{X}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ be a parametrized surface, for (u, v) belonging to some set of allowed parameters D in the u - v plane. Just like in the curve case, we can cut up the set D into N pieces P_1, \dots, P_N of area ΔA_i , where N is taken to be very large and each ΔA_i is small. Over a small range of the parameters, by the Locally Linear Principle, the surface can be approximated by

a part of a plane which is the parallelogram formed by the two tangent vectors \mathbf{X}_u and \mathbf{X}_v . If (u_i, v_i) is a point in the piece P_i then the area S_i of this piece is approximately $|\mathbf{X}_u \times \mathbf{X}_v| \Delta A_i$. Hence the total surface area S of this surface is

$$S = \lim_{N \rightarrow \infty} \sum_{i=1}^N |\mathbf{X}_u \times \mathbf{X}_v| \Delta A_i = \iint_D |\mathbf{X}_u \times \mathbf{X}_v| dudv$$

Again, if we are measuring some other quantity Q (like mass or charge) whose amount per unit area of the surface is the surface density function $\sigma(x, y, z)$, then the Density Principle tells us that the amount Q_i in the piece corresponding to the parameter range P_i is $\sigma(\mathbf{X}(u_i, v_i)) |\mathbf{X}_u \times \mathbf{X}_v| \Delta A_i$, and hence the total quantity Q is:

$$Q = \iint_D \sigma(\mathbf{X}(u, v)) |\mathbf{X}_u \times \mathbf{X}_v| dudv$$

We can also integrate vector fields over surfaces, if they are given an *orientation*, or a choice of unit normal vector field. This means we choose one of the two sides of the surfaces as “positive.” Again, suppose \mathbf{F} represents the velocity field of a fluid. This time we are interested in the rate of fluid passing through the surface per unit time. For a small piece P_i of the surface, which can be approximated by a part of a plane, the volume of fluid passing through per unit time equals the area S_i of the piece multiplied by the component of the velocity in the direction *normal* to the surface:

$$\begin{aligned} (\mathbf{F}(\mathbf{X}(u_i, v_i)) \cdot \mathbf{n}) S_i &= \mathbf{F}(\mathbf{X}(u_i, v_i)) \cdot \frac{\mathbf{X}_u \times \mathbf{X}_v}{|\mathbf{X}_u \times \mathbf{X}_v|} |\mathbf{X}_u \times \mathbf{X}_v| \Delta A_i \\ &= \mathbf{F}(\mathbf{X}(u_i, v_i)) \cdot (\mathbf{X}_u \times \mathbf{X}_v) \Delta A_i \end{aligned}$$

Adding up all the contributions over the whole surface, we get the *flux* of the vector field \mathbf{F} through the surface:

$$\text{Flux} = \iint_D \mathbf{F}(\mathbf{X}(u, v)) \cdot (\mathbf{X}_u \times \mathbf{X}_v) dudv$$

This measures the rate of fluid flowing through the surface. In Section ?? we will see how we can use the density principle to derive an expression for the flux per unit volume, a density function which we can integrate over solid objects. As in the curve case, it is computationally easier to do flux integrals rather than integrating functions over surfaces because there is no square root term which comes from $|\mathbf{X}_u \times \mathbf{X}_v|$ in the function case.

4. INTEGRALS OVER SOLID OBJECTS

A three-dimensional solid object is just a subset B of \mathbb{R}^3 . We can cut it up into N pieces P_1, \dots, P_N of volume ΔV_i . The volume V of B is thus

$$V = \lim_{N \rightarrow \infty} \sum_{i=1}^N \Delta V_i = \iiint_B dx dy dz$$

For a quantity Q (like mass or charge) whose amount per unit volume of the object is the density function $\sigma(x, y, z)$, then by the Density Principle the amount Q_i in

the piece corresponding to the parameter range P_i is $\sigma(\mathbf{r}_i)\Delta V_i$, where \mathbf{r}_i is a point in P_i and hence the total quantity Q is:

$$V = \iiint_B \sigma(x, y, z) dx dy dz$$

You might think that we should also be able to integrate a vector field \mathbf{F} over three dimensional regions, but we cannot. For line integrals, we take the dot product of \mathbf{F} with the unit tangent vector \mathbf{T} to the curve. For flux integrals, we take the dot product of \mathbf{F} with the unit normal vector \mathbf{n} to the surface. A solid three dimensional region does not have a natural vector at each point which we can use to take the dot product. There is also no physically meaningful notion we can assign to the integral of a vector field over a solid three dimensional region in \mathbf{R}^3 . You would learn more about why this is so if you took a course on calculus on manifolds.

5. THE FUNDAMENTAL THEOREMS OF CALCULUS IN TERMS OF DENSITIES

Let us recall the *density principle*: If we can measure some quantity Q of an object M , and $\sigma(x, y, z)$ is the density function of this quantity per unit size (length, area, or volume) then we can *recover* the density function $\sigma(\mathbf{r})$ from

$$\sigma(\mathbf{r}) = \lim_{M_r \rightarrow \mathbf{r}} \frac{Q_{M_r}}{S_{M_r}}$$

where M_r is a small piece of M containing the point \mathbf{r} . Here is a simple example: suppose we have an ordinary function $y = f(x)$ of one variable and are interested in the total *change* in the function between $x = a$ and $x = b$. Of course, this is just $f(b) - f(a)$. Since change *is* an additive quantity, there must be a density function $\sigma(x)$ which can be integrated over the curve (which here is the interval $[a, b]$ on the x -axis) to give the total change. We obtain this density function at $x = x_0$ by taking a small subinterval $[x_0, x_0 + \varepsilon]$ of $[a, b]$ and looking at

$$\sigma(x_0) = \lim_{\varepsilon \rightarrow 0} \frac{f(x_0 + \varepsilon) - f(x_0)}{\varepsilon} = f'(x_0)$$

Here the numerator is the quantity Q (change in the function) on the interval $[x_0, x_0 + \varepsilon]$ and the denominator is the size (length) of this interval. Thus we have shown that

$$\int_a^b f'(x) dx = f(b) - f(a)$$

which is just the original fundamental theorem of calculus.

Consider now the velocity field \mathbf{F} of a fluid. We would like to determine an expression for the density $\sigma(x, y, z)$ of outward flux per unit volume. That is, $\sigma(x, y, z)$ should be a function depending on the vector field \mathbf{F} so that

$$\iiint_B \sigma(x, y, z) dV = \text{Total Outward Flux} = \iint_{\partial B} \mathbf{F} \cdot \mathbf{n} dS$$

is the outward flux of the fluid through the boundary surface ∂B of the solid object B . From the density principle, this density function should be

$$\sigma(\mathbf{r}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^3} \iint_{\partial B(\mathbf{r}, \varepsilon)} \mathbf{F} \cdot \mathbf{n} dS$$

where $\partial B(\mathbf{r}, \varepsilon)$ is the boundary of a cube of side lengths ε with \mathbf{r} at one corner, and ε^3 is its volume. This integral breaks up into six different surface integrals, with the outward pointing normal in each case, and we have:

$$\iint_{\partial B(\mathbf{r}, \varepsilon)} \mathbf{F} \cdot \mathbf{n} dS \approx (F_1(\mathbf{r} + \varepsilon \mathbf{i}) - F_1(\mathbf{r}))\varepsilon^2 + F_2(\mathbf{r} + \varepsilon \mathbf{j}) - F_2(\mathbf{r})\varepsilon^2 + (F_3(\mathbf{r} + \varepsilon \mathbf{k}) - F_3(\mathbf{r}))\varepsilon^2$$

Here we have used the approximation that since the cube is very small, the components F_i of \mathbf{F} are approximately constant on each side of the cube. Now dividing by ε^3 and taking the limit as $\varepsilon \rightarrow 0$, we obtain the formula for $\sigma(\mathbf{r})$:

$$\sigma(x, y, z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

which is just the formula for the divergence $\nabla \cdot \mathbf{F}$ of the vector field \mathbf{F} . This way the divergence theorem is obtained by definition of the divergence as the outward flux density per unit volume. The tendency of a fluid to flow out of a closed object is thus the sum of the tendencies to flow out of the many small constituent pieces of that object.

We can similarly obtain Stokes' theorem. Here the situation will be slightly more complicated. Again \mathbf{F} is the velocity field of a fluid. Given a surface M with boundary curve ∂M , we would like to be able to calculate the circulation around the boundary curve by doing some kind of surface integral over the interior surface M . So we want to express the tendency of the fluid to circulate around the boundary curve as the sum of the tendencies to circulate around the many small constituent pieces of the interior surface. Since these small pieces are approximately planes, they have a normal direction (the normal to the surface at that point) and the right hand rule determines the orientation for circulation around the boundary of this piece. Thus any "circulation density per unit area" function that we want to reasonably define, should actually be a vector field, and its dot product with the unit normal \mathbf{n} at a point should be the density of circulation per unit area around a small loop centred at that point with \mathbf{n} as its normal.

Let $\boldsymbol{\sigma}(\mathbf{r})$ be the circulation density vector field. Then the density principle says that if \mathbf{n} is a unit vector,

$$\boldsymbol{\sigma}(\mathbf{r}) \cdot \mathbf{n} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_{\partial B(\mathbf{r}, \varepsilon)} \mathbf{F} \cdot \mathbf{T} ds$$

where this time $B(\mathbf{r}, \varepsilon)$ is a *square* of side lengths ε with \mathbf{r} at one corner and normal direction \mathbf{n} . So ε^2 is its area, and $\partial B(\mathbf{r}, \varepsilon)$ is its boundary curve. This line integral breaks up into four different line integrals, with orientation determined by the right hand rule. In the case $\mathbf{n} = \mathbf{k}$, for example, we have:

$$\int_{\partial B(\mathbf{r}, \varepsilon)} \mathbf{F} \cdot \mathbf{T} ds \approx (F_2(\mathbf{r} + \varepsilon \mathbf{i}) - F_2(\mathbf{r}))\varepsilon - (F_1(\mathbf{r} + \varepsilon \mathbf{j}) - F_1(\mathbf{r}))\varepsilon$$

Here we have used the approximation that since the square is very small, the components F_i of \mathbf{F} are approximately constant on each edge of the square. Now dividing by ε^2 and taking the limit as $\varepsilon \rightarrow 0$, we obtain the formula for $\boldsymbol{\sigma}(\mathbf{r}) \cdot \mathbf{k}$:

$$\boldsymbol{\sigma}(\mathbf{r}) \cdot \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

and similar calculation for $\mathbf{n} = \mathbf{i}$ or \mathbf{j} give us the expression for the circulation density vector field per unit area:

$$\boldsymbol{\sigma}(x, y, z) = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

which is just the formula for the curl $\nabla \times \mathbf{F}$ of the vector field \mathbf{F} . This way Stokes' theorem is obtained automatically by definition of the curl as the circulation density vector field per unit area.

Finally, let's obtain the fundamental theorem of line integrals in the same way and recover the characterization of the gradient in terms of directional derivatives. Consider a function $f(x, y, z)$ which can be thought of as the temperature at different points in space. If a particle travels along some path γ with initial point \mathbf{r}_i and final point \mathbf{r}_f , we would like to know the net *change* in temperature of the particle between the two positions. Of course, this change is just $f(\mathbf{r}_f) - f(\mathbf{r}_i)$. We would like to obtain this result by integrating some "temperature change density per unit length" over the path γ that the particle traverses.

We express the total change in temperature along the path as the sum of the changes along small constituent pieces of the path. Since these small pieces are approximately straight lines, they have a tangent direction. The change in a function of three variables of course depends on the direction we are moving in. Thus the "temperature change density per unit length" should be a vector field, and its dot product with the unit tangent \mathbf{T} at a point should be the density of function change per unit length for a small displacement in the direction of \mathbf{T} .

Let $\boldsymbol{\sigma}(\mathbf{r})$ be the temperature change density vector field. Then the density principle says that if \mathbf{T} is a unit vector,

$$\boldsymbol{\sigma}(\mathbf{r}) \cdot \mathbf{T} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (f(\mathbf{r} + \varepsilon \mathbf{T}) - f(\mathbf{r}))$$

where ε is the *length* of a small line segment in the \mathbf{T} direction. In the case $\mathbf{T} = \mathbf{k}$, for example, we have:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (f(x, y, z + \varepsilon) - f(x, y, z)) = \frac{\partial f}{\partial z}$$

and similar calculations for $\mathbf{T} = \mathbf{i}$ or \mathbf{j} give us the expression for the temperature change density vector field per unit length:

$$\boldsymbol{\sigma}(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

which is just the formula for the gradient ∇f of the function f . This way the fundamental theorem of line integrals is obtained automatically by definition of the gradient as the function change density vector field per unit length.

6. CONCLUSION

We have defined integration as the procedure of adding up lots of little additive quantities to obtain the total quantity. Then all the fundamental theorems of multivariable calculus arise naturally by considering various additive physical quantities and obtaining the appropriate "densities" that must be integrated to measure these quantities.