LINES AND PLANES IN $\mathbb{R}^3$

In this handout we will summarize the properties of the dot product and cross product and use them to present various descriptions of lines and planes in three dimensional space.

1. THE GEOMETRY OF THE DOT PRODUCT

Let $\mathbf{v} = <v_1, v_2, v_3>$ be a vector in $\mathbb{R}^3$. If $\mathbf{w} = <w_1, w_2, w_3>$ is another vector, then their *dot product*, denoted $\mathbf{v} \cdot \mathbf{w}$ is a *scalar* given by:

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= |\mathbf{v}||\mathbf{w}|\cos(\theta) \\ &= v_1 w_1 + v_2 w_2 + v_3 w_3 \end{aligned}$$

Here θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Now if $\mathbf{v}$ was a *unit vector*, then $\mathbf{v} \cdot \mathbf{w} = |\mathbf{w}|\cos(\theta)$ is the length of the projection of $\mathbf{w}$ along the vector $\mathbf{v}$. (If it is negative, its absolute value is the length of the projection, and the projection is in the opposite direction of $\mathbf{v}$.) If we keep the length $|\mathbf{w}|$ constant but move the vector $\mathbf{w}$ around, the maximal length of the projection occurs when $\cos(\theta) = \pm 1$, which corresponds to $\theta = 0$ or $\theta = \pi$. That is, when $\mathbf{w}$ is *parallel* to $\mathbf{v}$. The length is zero when $\mathbf{w}$ and $\mathbf{v}$ are *orthogonal* (perpendicular), which corresponds to $\theta = \frac{\pi}{2}$. The set of vectors which are orthogonal to $\mathbf{v}$ is a *plane* and $\mathbf{v}$ is called a *normal* vector to the plane. A part of this plane is shown in Figure 1 as a disc. The vector $\mathbf{u}$ is in the plane and $\mathbf{u} \cdot \mathbf{v} = 0$. Hence the dot product measures how close a vector is to being parallel to another vector.
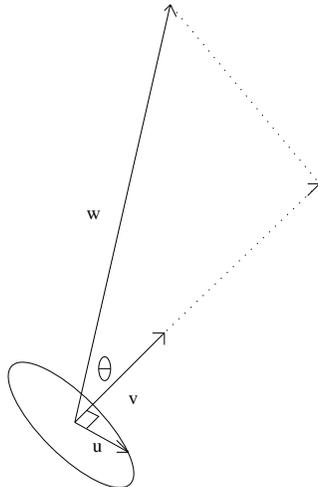

Figure 1: The Dot Product

FIGURE 1. The Dot Product

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2. The Geometry of the Cross Product

The *cross product* of two vectors $\mathbf{v}$ and $\mathbf{w}$ is another *vector*, denoted $\mathbf{v} \times \mathbf{w}$ and is given by:

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= \ <v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1> \\ |\mathbf{v} \times \mathbf{w}| &= \ |\mathbf{v}||\mathbf{w}|\sin(\theta) \end{aligned}$$

The *direction* of $\mathbf{v} \times \mathbf{w}$ is given by the right hand rule: $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ (and so is orthogonal to the *plane* determined by $\mathbf{v}$ and $\mathbf{w}$) and points in the direction of your thumb if you point the four fingers of your right hand in the $\mathbf{v}$ direction and curl them toward the $\mathbf{w}$ direction.

The cross product is *anti-commutative*: $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$. Also, when $\mathbf{w}$ is parallel to $\mathbf{v}$, then $\theta = 0$ or $\theta = \pi$ and $\mathbf{v} \times \mathbf{w} = \mathbf{0}$. So the dot product equals the scalar 0 on perpendicular vectors but the cross product equals the *vector* $\mathbf{0}$ on *parallel* vectors.

The length $|\mathbf{v} \times \mathbf{w}|$ of the cross product equals the *area* of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. If we keep the length $|\mathbf{w}|$ constant but move the vector $\mathbf{w}$ around, the maximal area occurs when $\sin(\theta) = 1$, which corresponds to $\theta = \frac{\pi}{2}$. That is, when $\mathbf{w}$ is *orthogonal* to $\mathbf{v}$. The area is zero when $\mathbf{w}$ and $\mathbf{v}$ are *parallel*, since their cross product is $\mathbf{0}$ in that case. The set of vectors which are parallel to $\mathbf{v}$ is a *line* and $\mathbf{v}$ is called a *direction vector* of the line. This can be seen in Figure 2.
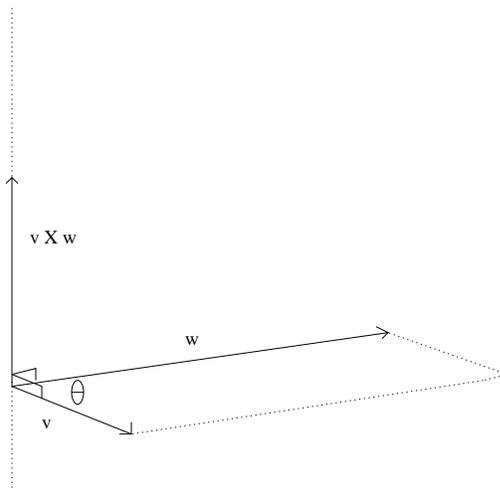

Figure 2: The Cross Product

FIGURE 2. The Cross Product

3. Lines in $\mathbb{R}^3$

Suppose we had an infinite straight line in $\mathbb{R}^3$. Think of it as an infinite ruler. This ruler has some scale on it, which is determined by where the origin is, and by the distance between successive integer units of length. (Once we pick our origin, we also have to choose which side of the origin will be positive and which will be negative. So it is more like a thermometer than a ruler.) We can describe the line

by giving all the points on the line. Let us label the origin ("zero temperature") of our infinite thermometer by **p**, which is a vector from the origin of $\mathbf{R}^3$ to the origin of the thermometer. Let the vector **v** be the vector from the origin of the thermometer to the position which is at "temperature" $+1$ on the thermometer. (See Figure 3.) Now to get from the origin of $\mathbb{R}^3$, to the point whose "temperature" on our thermometer is t first we get to the origin of our thermometer which is a displacement **p**, then we need to add to this a displacement of $t\mathbf{v}$. Hence we see that the vector $\mathbf{p} + t\mathbf{v}$ has its tip on our line, if its tail is at $(0, 0, 0)$. Also, as we vary t over all possible real numbers, positive, negative, and zero, we get *all* the points on this infinite line. This is called a set of *parametric* equations for the line, and t is called the parameter.
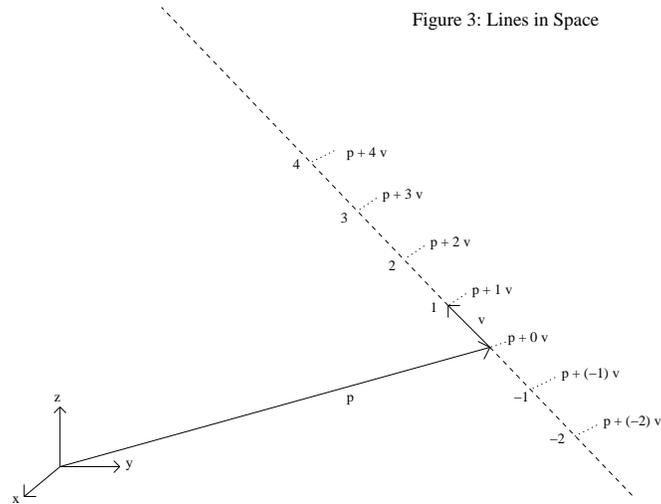

Figure 3: Lines in Space

FIGURE 3. The Line Through **p** with Direction Vector **v**

If we write $\mathbf{p} = <x_0, y_0, z_0>$ and $\mathbf{v} = <v_1, v_2, v_3>$, then the parametric equations correspond to three functions of t:

$$
\begin{aligned}
x &= x_0 + v_1 t \\
y &= y_0 + v_2 t \\
z &= z_0 + v_3 t
\end{aligned}
$$

We get the whole line for $-\infty < t < \infty$, and we can describe just a segment of the line by restricting the parameter t to satisfy $a \le t \le b$ for some endpoints a and b.

Parametric equations for a line are *not unique*. This is a mathematician's way of saying that there are infinitely many ways of describing the same straight line in this way and they are *all* correct. This is easily seen: in the above description, we can choose *any* point **p** on the line and *any* vector **v** in the direction of the line. In the thermometer analogy, choosing a different **p** corresponds to choosing a different point on our "infinite thermometer" to correspond to zero temperature, and changing the length of **v** corresponds to changing the scale (distance between successive degrees). We can also change the direction of **v**, which corresponds to reversing the positive and negative directions of the thermometer. But it is always still the same infinite line.

Before we move on to planes, let us rewrite the equations for a line using the cross product, and eliminating the parameter t in the process. Let $\mathbf{r} =< x, y, z >$ be a varying point on the line. Then $\mathbf{r} - \mathbf{p} = t\mathbf{v}$, so it is parallel to $\mathbf{v}$ and so we can write:

$$(\mathbf{r} - \mathbf{p}) \times \mathbf{v} = \mathbf{0}$$

This is sometimes called the *vector equation* for a line. It says that if $\mathbf{p}$ is a fixed point on the line, and $\mathbf{v}$ is a direction vector for the line, then all other points $\mathbf{r}$ on the line satisfy that equation.

4. PLANES IN $\mathbb{R}^3$

Now suppose we had an infinite plane in $\mathbb{R}^3$. Think of it as an infinite map of the earth. (Pretend the earth was flat and was infinite). We could pick an origin on this map and draw Cartesian x- and y- axes to describe all the points on it. (Like latitude and longitude on a flat map). You can also think of it as drawing your map on graph paper. But we don't need to make the graph paper out of squares. We can use any parallelogram. On our parallelogram graph paper, we can pick an origin $(0, 0)$ which corresponds to the vertex of one parallelogram, and label the opposite vertex of the same parallelogram as the point $(1, 1)$. This corresponds to picking two vectors $\mathbf{v}$ and $\mathbf{w}$ on the plane both starting at the origin of our plane with $(1, 1)$ corresponding to $\mathbf{v} + \mathbf{w}$. Let us label the origin of our infinite plane by $\mathbf{p}$, which is a vector from the origin of $\mathbf{R}^3$ to the origin of the plane. (See Figure 4.) Now to get from the origin of $\mathbb{R}^3$, to the point whose coordinates on our "graph paper" are (s, t) first we get to the origin of our plane which is a displacement $\mathbf{p}$, then we need to add to this a displacement of $s\mathbf{v} + t\mathbf{w}$. Hence we see that the vector $\mathbf{p} + s\mathbf{v} + t\mathbf{w}$ has its tip on our plane, if its tail is at $(0, 0, 0)$. Also, as we vary s and t over all possible real numbers, positive, negative, and zero, we get *all* the points on this infinite plane. This is called a set of *parametric* equations for the plane, and s and t are called the parameters.
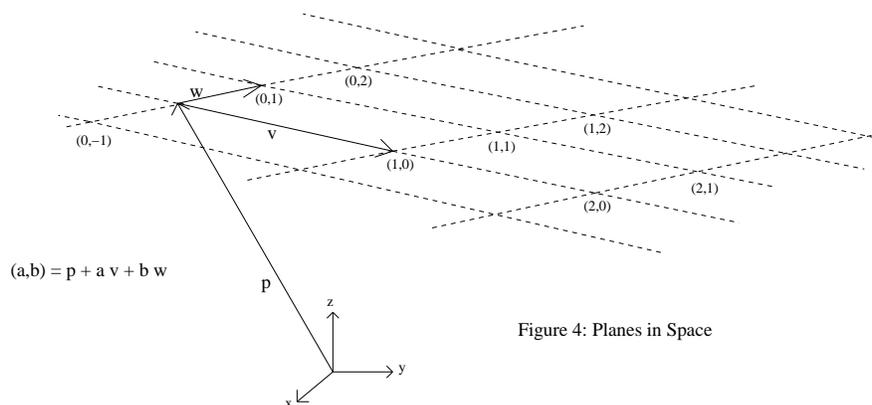

Figure 4: Planes in Space

FIGURE 4. The Plane Through $\mathbf{p}$ Containing The Vectors $\mathbf{v}$ and $\mathbf{w}$

If we write $\mathbf{p} =< x_0, y_0, z_0 >$, $\mathbf{v} =< v_1, v_2, v_3 >$, and $w =< w_1, w_2, w_3 >$ then the parametric equations correspond to three functions of s and t:

$$
\begin{aligned}
x &= x_0 + v_1 s + w_1 t \\
y &= y_0 + v_2 s + w_2 t \\
z &= z_0 + v_3 s + w_3 t
\end{aligned}
$$

We get the whole plane for $-\infty < s, t < \infty$, and we can describe just a portion of the plane by restricting the parameters s,t to satisfy $a \leq s \leq b$ and $c \leq t \leq d$ for some $a < b$ and $c < d$.

Just like in the case for a line, parametric equations for a plane are *not unique*. There are infinitely many ways of describing the same plane in this way and they are *all* correct. This is easily seen: in the above description, we can choose *any* point $\mathbf{p}$ on the plane and *any* vectors $\mathbf{v}$ and $\mathbf{w}$ contained in the plane. In the map-graph paper analogy, choosing a different $\mathbf{p}$ corresponds to choosing a different origin, and choosing different vectors $\mathbf{v}$ and $\mathbf{w}$ corresponds to a different size and shape parallelogram. But it is always still the same infinite plane.

Now since $\mathbf{v}$ and $\mathbf{w}$ are both in the plane, the vector $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ is *orthogonal* to the plane, and is called a *normal vector* for the plane. Now let us rewrite the equations for a plane using the cross product, and eliminating the parameters s and t in the process. Let $\mathbf{r} =< x, y, z >$ be a varying point on the plane. Then $\mathbf{r} - \mathbf{p} = s\mathbf{v} + t\mathbf{w}$, so it is perpendicular to $\mathbf{n}$ (since both $\mathbf{v}$ and $\mathbf{w}$ are) and so we can write:

$$(\mathbf{r} - \mathbf{p}) \cdot \mathbf{n} = 0$$

This is sometimes called the *vector equation* for a plane. It says that if $\mathbf{p}$ is a fixed point on the line, and $\mathbf{n}$ is a normal vector for the plane, then all other points $\mathbf{r}$ on the plane satisfy that equation. This equation is often written in another *scalar* form: If $\mathbf{n} =< a, b, c >$, then we can write:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

If we collect together all the constant terms on to the right hand side, we get:

$$ax + by + cz = d$$

for some constant d. The points (x, y, z) which satisfy the equation are all on a plane which has $\mathbf{n} =< a, b, c >$ as a normal vector. (Any non-zero multiple of $\mathbf{n}$ is another valid normal vector).

5. SUMMARY

We have seen two ways to describe lines and two ways to describe planes: The parametric equations give all points on the line (plane) by varying the one (two) parameter(s) over all possible real values. We say a line is *one-dimensional* because it only requires *one* parameter to describe where you are on it. Any line basically looks just like a copy of the real line $\mathbb{R}$. Similarly a plane is *two-dimensional* because we need *two* parameters to describe it completely. Any plane looks just like a copy of the ordinary plane $\mathbb{R}^2$.

Very soon in this course we will talk about curves and surfaces, which are one and two dimensional objects that aren't "flat" lines or planes, but which curve around in space. However, on a very small scale (think of putting them under a microscope) the curves "look like" lines and the surfaces "look like" planes.

The other way we described lines and planes was using the cross product and the dot product. Since all the vectors which are parallel to a fixed vector $\mathbf{v}$ have cross product $\mathbf{0}$ with $\mathbf{v}$, the set of all such vectors is a line with direction vector $\mathbf{v}$. Similarly since all the vectors which are orthogonal to a fixed vector $\mathbf{n}$ have dot product 0 with $\mathbf{n}$, the set of all such vectors is a plane with normal vector $\mathbf{n}$.

Although they won't come up in this course, here are some interesting remarks: The concept of dot product makes sense in *any* dimension. That is, we can talk about vectors in $\mathbb{R}^n$ for any positive integer n and define the dot product and use it to measure orthogonality. However, it turns out that the cross product only makes sense in $\mathbb{R}^3$ and $\mathbb{R}^7$. (This is *not at all* obvious). The analogues of the above descriptions of lines and planes become a little more complicated in higher dimensions. The parametric equation description for both lines and planes remains true in any dimension. The "vector equation" for a plane which was $(\mathbf{r} - \mathbf{p}) \cdot \mathbf{n} = 0$ in $\mathbb{R}^3$ now describes a "flat" $(n-1)$-dimensional object that "looks like" $\mathbb{R}^{n-1}$. These are called "hyperplanes" and these are the easiest objects to describe. When $n = 3$, we have $n - 1 = 2$ and the hyperplanes are two dimensional. If $n > 3$, two-dimensional planes get much harder to describe non-parametrically.