

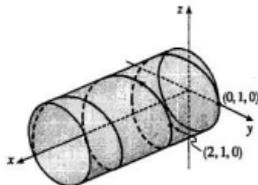
7. (a) The unit normal vector: $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$. The binormal vector: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.
 (b) See the discussion preceding Example 7 in Section 10.3.
8. (a) If $\mathbf{r}(t)$ is the position vector of the particle on the space curve, the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$, the speed is given by $|\mathbf{v}(t)|$, and the acceleration $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.
 (b) $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$ where $a_T = v'$ and $a_N = \kappa v^2$.
9. See the statement of Kepler's Laws on page 731.
10. See the discussion on pages 736 and 737.

▲ TRUE-FALSE QUIZ ▲

1. True. If we reparametrize the curve by replacing $u = t^3$, we have $\mathbf{r}(u) = u \mathbf{i} + 2u \mathbf{j} + 3u \mathbf{k}$, which is a line through the origin with direction vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
2. True. $\mathbf{r}'(t) = \langle 1, 3t^2, 5t^4 \rangle$ is continuous for all t (since its component functions are each continuous) and since $x'(t) = 1$, we have $\mathbf{r}'(t) \neq \mathbf{0}$ for all t .
3. False. $\mathbf{r}'(t) = \langle -\sin t, 2t, 4t^3 \rangle$, and since $\mathbf{r}'(0) = \langle 0, 0, 0 \rangle = \mathbf{0}$, the curve is not smooth.
4. True. See Theorem 10.2.2.
5. False. By Formula 5 of Theorem 10.2.3, $\frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.
6. False. For example, let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. Then $|\mathbf{r}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1 \Rightarrow \frac{d}{dt} |\mathbf{r}(t)| = 0$, but $|\mathbf{r}'(t)| = | \langle -\sin t, \cos t \rangle | = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$.
7. False. κ is the magnitude of the rate of change of the unit tangent vector \mathbf{T} with respect to arc length s , not with respect to t .
8. False. The binormal vector, by the definition given in Section 10.3, is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = -[\mathbf{N}(t) \times \mathbf{T}(t)]$.
9. True. See the discussion preceding Example 7 in Section 10.3.
10. False. For example, $\mathbf{r}_1(t) = \langle t, t \rangle$ and $\mathbf{r}_2(t) = \langle 2t, 2t \rangle$ both represent the same plane curve (the line $y = x$), but the tangent vector $\mathbf{r}'_1(t) = \langle 1, 1 \rangle$ for all t , while $\mathbf{r}'_2(t) = \langle 2, 2 \rangle$. In fact, different parametrizations give parallel tangent vectors at a point, but their magnitudes may differ.

◆ EXERCISES ◆

1. (a) The corresponding parametric equations for the curve are $x = t$,
 $y = \cos \pi t$, $z = \sin \pi t$. Since $y^2 + z^2 = 1$, the curve is contained
 in a circular cylinder with axis the x -axis. Since $x = t$, the curve is
 a helix.



- (b) $\mathbf{r}(t) = t \mathbf{i} + \cos \pi t \mathbf{j} + \sin \pi t \mathbf{k} \Rightarrow \mathbf{r}'(t) = \mathbf{i} - \pi \sin \pi t \mathbf{j} + \pi \cos \pi t \mathbf{k} \Rightarrow$
 $\mathbf{r}''(t) = -\pi^2 \cos \pi t \mathbf{j} - \pi^2 \sin \pi t \mathbf{k}$

2. (a) The expressions $\sqrt{2-t}$, $(e^t - 1)/t$, and $\ln(t+1)$ are all defined when $2-t \geq 0 \Rightarrow t \leq 2$, $t \neq 0$, and $t+1 > 0 \Rightarrow t > -1$. Thus the domain of \mathbf{r} is $(-1, 0) \cup (0, 2]$.

$$(b) \lim_{t \rightarrow 0} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow 0} \sqrt{2-t}, \lim_{t \rightarrow 0} \frac{e^t - 1}{t}, \lim_{t \rightarrow 0} \ln(t+1) \right\rangle = \left\langle \sqrt{2-0}, \lim_{t \rightarrow 0} \frac{e^t}{1}, \ln(0+1) \right\rangle = \langle \sqrt{2}, 1, 0 \rangle$$

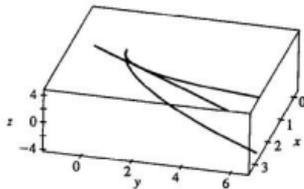
(using l'Hospital's Rule in the y -component).

$$(c) \mathbf{r}'(t) = \left\langle \frac{d}{dt} \sqrt{2-t}, \frac{d}{dt} \frac{e^t - 1}{t}, \frac{d}{dt} \ln(t+1) \right\rangle = \left\langle -\frac{1}{2\sqrt{2-t}}, \frac{te^t - e^t + 1}{t^2}, \frac{1}{t+1} \right\rangle$$

3. The projection of the curve C of intersection onto the xy -plane is the circle $x^2 + y^2 = 16$, $z = 0$. So we can write $x = 4 \cos t$, $y = 4 \sin t$, $0 \leq t \leq 2\pi$. From the equation of the plane, we have $z = 5 - x = 5 - 4 \cos t$, so parametric equations for C are $x = 4 \cos t$, $y = 4 \sin t$, $z = 5 - 4 \cos t$, $0 \leq t \leq 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j} + (5 - 4 \cos t) \mathbf{k}$, $0 \leq t \leq 2\pi$.

4. The curve is given by $\mathbf{r}(t) = \langle t^2, t^4, t^3 \rangle$, so $\mathbf{r}'(t) = \langle 2t, 4t^3, 3t^2 \rangle$.

The point $(1, 1, 1)$ corresponds to $t = 1$, so the tangent vector there is $\mathbf{r}'(1) = \langle 2, 4, 3 \rangle$. Then the tangent line has direction vector $\langle 2, 4, 3 \rangle$ and includes the point $(1, 1, 1)$, so parametric equations are $x = 1 + 2t$, $y = 1 + 4t$, $z = 1 + 3t$.



$$\begin{aligned} 5. \int_0^1 (t^2 \mathbf{i} + t \cos \pi t \mathbf{j} + \sin \pi t \mathbf{k}) dt &= \left(\int_0^1 t^2 dt \right) \mathbf{i} + \left(\int_0^1 t \cos \pi t dt \right) \mathbf{j} + \left(\int_0^1 \sin \pi t dt \right) \mathbf{k} \\ &= \left[\frac{1}{3} t^3 \right]_0^1 \mathbf{i} + \left(\frac{1}{\pi} \sin \pi t \right)_0^1 - \int_0^1 \frac{1}{\pi} \sin \pi t dt \mathbf{j} + \left[-\frac{1}{\pi} \cos \pi t \right]_0^1 \mathbf{k} \\ &= \frac{1}{3} \mathbf{i} + \left[\frac{1}{\pi^2} \cos \pi t \right]_0^1 \mathbf{j} + \frac{2}{\pi} \mathbf{k} = \frac{1}{3} \mathbf{i} - \frac{2}{\pi^2} \mathbf{j} + \frac{2}{\pi} \mathbf{k} \end{aligned}$$

where we integrated by parts in the y -component.

6. (a) C intersects the xz -plane where $y = 0 \Rightarrow 2t - 1 = 0 \Rightarrow t = \frac{1}{2}$, so the point is

$$\left(2 - \left(\frac{1}{2}\right)^3, 0, \ln \frac{1}{2} \right) = \left(\frac{15}{8}, 0, -\ln 2 \right).$$

- (b) The curve is given by $\mathbf{r}(t) = \langle 2 - t^3, 2t - 1, \ln t \rangle$, so $\mathbf{r}'(t) = \langle -3t^2, 2, 1/t \rangle$. The point $(1, 1, 0)$ corresponds to $t = 1$, so the tangent vector there is $\mathbf{r}'(1) = \langle -3, 2, 1 \rangle$. Then the tangent line has direction vector $\langle -3, 2, 1 \rangle$ and includes the point $(1, 1, 0)$, so parametric equations are $x = 1 - 3t$, $y = 1 + 2t$, $z = t$.

- (c) The normal plane has normal vector $\mathbf{r}'(1) = \langle -3, 2, 1 \rangle$ and equation $-3(x-1) + 2(y-1) + z = 0$ or $3x - 2y - z = 1$.

7. $t = 1$ at $(1, 4, 2)$ and $t = 4$ at $(2, 1, 17)$, so

$$\begin{aligned} L &= \int_1^4 \sqrt{\frac{1}{4t} + \frac{16}{t^4} + 4t^2} dt \\ &\approx \frac{4-1}{3 \cdot 4} \left[\sqrt{\frac{1}{4} + 16 + 4} + 4 \cdot \sqrt{\frac{1}{4 \cdot \frac{7}{4}} + \frac{16}{(\frac{7}{4})^4} + 4 \left(\frac{7}{4}\right)^2} + 2 \cdot \sqrt{\frac{1}{4 \cdot \frac{10}{4}} + \frac{16}{(\frac{10}{4})^4} + 4 \left(\frac{10}{4}\right)^2} \right. \\ &\quad \left. + 4 \cdot \sqrt{\frac{1}{4 \cdot \frac{13}{4}} + \frac{16}{(\frac{13}{4})^4} + 4 \left(\frac{13}{4}\right)^2} + \sqrt{\frac{1}{4 \cdot 4} + \frac{16}{4^4} + 4 \cdot 4^2} \right] \end{aligned}$$

$$\approx 15.9241$$

8. $\mathbf{r}'(t) = \langle 3t^{1/2}, -2 \sin 2t, 2 \cos 2t \rangle$, $|\mathbf{r}'(t)| = \sqrt{9t + 4(\sin^2 2t + \cos^2 2t)} = \sqrt{9t + 4}$. Thus

$$L = \int_0^1 \sqrt{9t+4} dt = \int_4^{13} \frac{1}{9} u^{1/2} du = \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_4^{13} = \frac{2}{27} (13^{3/2} - 8).$$

9. The angle of intersection of the two curves, θ , is the angle between their respective tangents at the point of intersection. For both curves the point $(1, 0, 0)$ occurs when $t = 0$. $\mathbf{r}'_1(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}'_1(0) = \mathbf{j} + \mathbf{k}$ and $\mathbf{r}'_2(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \Rightarrow \mathbf{r}'_2(0) = \mathbf{i}$. $\mathbf{r}'_1(0) \cdot \mathbf{r}'_2(0) = (\mathbf{j} + \mathbf{k}) \cdot \mathbf{i} = 0$. Therefore, the curves intersect in a right angle, that is, $\theta = \frac{\pi}{2}$.

10. The parametric value corresponding to the point $(1, 0, 1)$ is $t = 0$.

$$\mathbf{r}'(t) = e^t \mathbf{i} + e^t(\cos t + \sin t)\mathbf{j} + e^t(\cos t - \sin t)\mathbf{k}$$

$$\Rightarrow |\mathbf{r}'(t)| = e^t \sqrt{1 + (\cos t + \sin t)^2 + (\cos t - \sin t)^2} = \sqrt{3} e^t$$

$$\text{and } s(t) = \int_0^t e^u \sqrt{3} du = \sqrt{3}(e^t - 1) \Rightarrow t = \ln\left(1 + \frac{1}{\sqrt{3}}s\right). \text{ Therefore,}$$

$$\mathbf{r}(t(s)) = \left(1 + \frac{1}{\sqrt{3}}s\right) \mathbf{i} + \left(1 + \frac{1}{\sqrt{3}}s\right) \sin \ln\left(1 + \frac{1}{\sqrt{3}}s\right) \mathbf{j} + \left(1 + \frac{1}{\sqrt{3}}s\right) \cos \ln\left(1 + \frac{1}{\sqrt{3}}s\right) \mathbf{k}.$$

$$11. (a) \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle t^2, t, 1 \rangle}{|\langle t^2, t, 1 \rangle|} = \frac{\langle t^2, t, 1 \rangle}{\sqrt{t^4 + t^2 + 1}}$$

$$(b) \mathbf{T}'(t) = -\frac{1}{2}(t^4 + t^2 + 1)^{-3/2} (4t^3 + 2t) \langle t^2, t, 1 \rangle + (t^4 + t^2 + 1)^{-1/2} \langle 2t, 1, 0 \rangle$$

$$= \frac{-2t^3 - t}{(t^4 + t^2 + 1)^{3/2}} \langle t^2, t, 1 \rangle + \frac{1}{(t^4 + t^2 + 1)^{1/2}} \langle 2t, 1, 0 \rangle$$

$$= \frac{\langle -2t^5 - t^3, -2t^4 - t^2, -2t^3 - t \rangle + \langle 2t^5 + 2t^3 + 2t, t^4 + t^2 + 1, 0 \rangle}{(t^4 + t^2 + 1)^{3/2}}$$

$$= \frac{\langle 2t, -t^4 + 1, -2t^3 - t \rangle}{(t^4 + t^2 + 1)^{3/2}}$$

$$|\mathbf{T}'(t)| = \frac{\sqrt{4t^2 + t^8 - 2t^4 + 1 + 4t^6 + 4t^4 + t^2}}{(t^4 + t^2 + 1)^{3/2}} = \frac{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}{(t^4 + t^2 + 1)^{3/2}}, \text{ and}$$

$$\mathbf{N}(t) = \frac{\langle 2t, 1 - t^4, -2t^3 - t \rangle}{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}.$$

$$(c) \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}{(t^4 + t^2 + 1)^2}$$

12. Using Exercise 10.3.32, we have $\mathbf{r}'(t) = \langle -3 \sin t, 4 \cos t \rangle$, $\mathbf{r}''(t) = \langle -3 \cos t, -4 \sin t \rangle$,

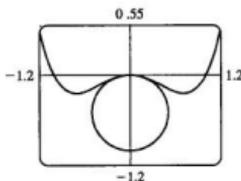
$$|\mathbf{r}'(t)|^3 = \left(\sqrt{9 \sin^2 t + 16 \cos^2 t}\right)^3 \text{ and then}$$

$$\kappa(t) = \frac{|(-3 \sin t)(-4 \sin t) - (4 \cos t)(-3 \cos t)|}{(9 \sin^2 t + 16 \cos^2 t)^{3/2}} = \frac{12}{(9 \sin^2 t + 16 \cos^2 t)^{3/2}}.$$

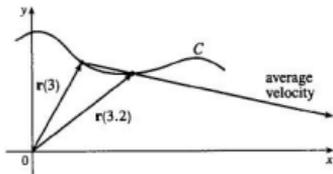
At $(3, 0)$, $t = 0$ and $\kappa(0) = 12/(16)^{3/2} = \frac{12}{64} = \frac{3}{16}$. At $(0, 4)$, $t = \frac{\pi}{2}$ and $\kappa(\frac{\pi}{2}) = 12/9^{3/2} = \frac{12}{27} = \frac{4}{9}$.

$$13. y' = 4x^3, y'' = 12x^2 \text{ and } \kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|12x^2|}{(1 + 16x^6)^{3/2}}, \text{ so } \kappa(1) = \frac{12}{17^{3/2}}.$$

14. $\kappa(x) = \frac{|12x^2 - 2|}{[1 + (4x^3 - 2x)^2]^{3/2}} \Rightarrow \kappa(0) = 2$. So the osculating circle has radius $\frac{1}{2}$ and center $(0, -\frac{1}{2})$. Thus its equation is $x^2 + (y + \frac{1}{2})^2 = \frac{1}{4}$.



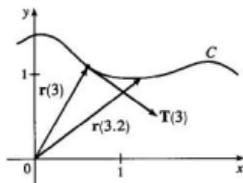
15. $\mathbf{r}(t) = \langle \sin 2t, t, \cos 2t \rangle \Rightarrow \mathbf{r}'(t) = \langle 2 \cos 2t, 1, -2 \sin 2t \rangle \Rightarrow \mathbf{T}(t) = \frac{1}{\sqrt{5}} \langle 2 \cos 2t, 1, -2 \sin 2t \rangle \Rightarrow \mathbf{T}'(t) = \frac{1}{\sqrt{5}} \langle -4 \sin 2t, 0, -4 \cos 2t \rangle \Rightarrow \mathbf{N}(t) = \langle -\sin 2t, 0, -\cos 2t \rangle$. So $\mathbf{N} = \mathbf{N}(\pi) = \langle 0, 0, -1 \rangle$ and $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{5}} \langle -1, 2, 0 \rangle$. So a normal to the osculating plane is $\langle -1, 2, 0 \rangle$ and an equation is $-1(x-0) + 2(y-\pi) + 0(z-1) = 0$ or $x - 2y + 2\pi = 0$.
16. (a) The average velocity over $[3, 3.2]$ is given by $\frac{\mathbf{r}(3.2) - \mathbf{r}(3)}{3.2 - 3} = 5[\mathbf{r}(3.2) - \mathbf{r}(3)]$, so we draw a vector with the same direction but 5 times the length of the vector $[\mathbf{r}(3.2) - \mathbf{r}(3)]$.



$$(b) \mathbf{v}(3) = \mathbf{r}'(3) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(3+h) - \mathbf{r}(3)}{h}$$

$$(c) \mathbf{T}(3) = \frac{\mathbf{r}'(3)}{|\mathbf{r}'(3)|}, \text{ a unit vector in the same direction as } \mathbf{r}'(3), \text{ that is,}$$

parallel to the tangent line to the curve at $\mathbf{r}(3)$, pointing in the direction corresponding to increasing t , and with length 1.



17. $\mathbf{r}(t) = t \ln t \mathbf{i} + t \mathbf{j} + e^{-t} \mathbf{k}$, $\mathbf{v}(t) = \mathbf{r}'(t) = (1 + \ln t) \mathbf{i} + \mathbf{j} - e^{-t} \mathbf{k}$,
 $|\mathbf{v}(t)| = \sqrt{(1 + \ln t)^2 + 1^2 + (-e^{-t})^2} = \sqrt{2 + 2 \ln t + (\ln t)^2 + e^{-2t}}$, $\mathbf{a}(t) = \mathbf{v}'(t) = \frac{1}{t} \mathbf{i} + e^{-t} \mathbf{k}$.
18. $\mathbf{v}(t) = \int \mathbf{a}(t) dt = \int (6t \mathbf{i} + 12t^2 \mathbf{j} - 6t \mathbf{k}) dt = 3t^2 \mathbf{i} + 4t^3 \mathbf{j} - 3t^2 \mathbf{k} + \mathbf{C}$, but $\mathbf{i} - \mathbf{j} + 3\mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{C}$, so $\mathbf{C} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v}(t) = (3t^2 + 1) \mathbf{i} + (4t^3 - 1) \mathbf{j} + (3 - 3t^2) \mathbf{k}$.
 $\mathbf{r}(t) = \int \mathbf{v}(t) dt = (t^3 + t) \mathbf{i} + (t^4 - t) \mathbf{j} + (3t - t^3) \mathbf{k} + \mathbf{D}$. But $\mathbf{r}(0) = \mathbf{0}$, so $\mathbf{D} = \mathbf{0}$ and
 $\mathbf{r}(t) = (t^3 + t) \mathbf{i} + (t^4 - t) \mathbf{j} + (3t - t^3) \mathbf{k}$.

19. We set up the axes so that the shot leaves the athlete's hand 7 ft above the origin. Then we are given $\mathbf{r}(0) = 7\mathbf{j}$, $|\mathbf{v}(0)| = 43$ ft/s, and $\mathbf{v}(0)$ has direction given by a 45° angle of elevation. Then a unit vector in the direction of $\mathbf{v}(0)$ is $\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \Rightarrow \mathbf{v}(0) = \frac{43}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$. Assuming air resistance is negligible, the only external force is due to gravity, so as in Example 10.4.5 we have $\mathbf{a} = -g\mathbf{j}$ where here $g \approx 32$ ft/s². Since $\mathbf{v}'(t) = \mathbf{a}(t)$, we integrate, giving $\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{C}$ where $\mathbf{C} = \mathbf{v}(0) = \frac{43}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \Rightarrow \mathbf{v}(t) = \frac{43}{\sqrt{2}}\mathbf{i} + \left(\frac{43}{\sqrt{2}} - gt\right)\mathbf{j}$. Since $\mathbf{r}'(t) = \mathbf{v}(t)$ we integrate again, so $\mathbf{r}(t) = \frac{43}{\sqrt{2}}t\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{D}$. But $\mathbf{D} = \mathbf{r}(0) = 7\mathbf{j} \Rightarrow \mathbf{r}(t) = \frac{43}{\sqrt{2}}t\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7\right)\mathbf{j}$.

(a) At 2 seconds, the shot is at $\mathbf{r}(2) = \frac{43}{\sqrt{2}}(2)\mathbf{i} + \left(\frac{43}{\sqrt{2}}(2) - \frac{1}{2}g(2)^2 + 7\right)\mathbf{j} \approx 60.8\mathbf{i} + 3.8\mathbf{j}$, so the shot is about 3.8 ft above the ground, at a horizontal distance of 60.8 ft from the athlete.

(b) The shot reaches its maximum height when the vertical component of velocity is 0: $\frac{43}{\sqrt{2}} - gt = 0 \Rightarrow t = \frac{43}{\sqrt{2}g} \approx 0.95$ s. Then $\mathbf{r}(0.95) \approx 28.9\mathbf{i} + 21.4\mathbf{j}$, so the maximum height is approximately 21.4 ft.

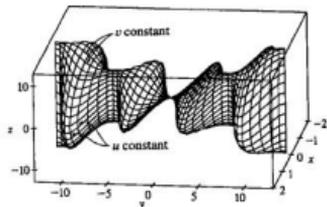
(c) The shot hits the ground when the vertical component of $\mathbf{r}(t)$ is 0, so $\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7 = 0 \Rightarrow -16t^2 + \frac{43}{\sqrt{2}}t + 7 = 0 \Rightarrow t \approx 2.11$ s. $\mathbf{r}(2.11) \approx 64.2\mathbf{i} - 0.08\mathbf{j}$, thus the shot lands approximately 64.2 ft from the athlete.

20. $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k}$, $\mathbf{r}''(t) = 2\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1 + 4 + 4t^2} = \sqrt{4t^2 + 5}$.

Then $a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{4t}{\sqrt{4t^2 + 5}}$ and $a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{|4\mathbf{i} - 2\mathbf{j}|}{\sqrt{4t^2 + 5}} = \frac{2\sqrt{5}}{\sqrt{4t^2 + 5}}$.

21. From Example 4 in Section 10.5, a parametric representation of the sphere $x^2 + y^2 + z^2 = 4$ is $x = 2 \sin \phi \cos \theta$, $y = 2 \sin \phi \sin \theta$, $z = 2 \cos \phi$ with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. We can restrict the surface to that portion between the planes $z = 1$ and $z = -1$ by restricting $-1 \leq z \leq 1 \Rightarrow -1 \leq 2 \cos \phi \leq 1 \Rightarrow \frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}$.

22. $x = (1 - \cos u) \sin v$, $y = u$, $z = (u - \sin u) \cos v$. We plot the portion of the surface corresponding to the parametric domain $-4\pi \leq u \leq 4\pi$, $0 \leq v \leq 2\pi$. If u is held constant, x and z give the equation of an ellipse in the plane $y = u$, thus the grid curves are the vertically oriented ellipses we see. Note that when $u = 2k\pi$, k an integer, we have $x = 0$ and z ranges from $-2k\pi$ to $2k\pi$, representing a line segment parallel to the z -axis in the plane $y = 2k\pi$. These are the vertical "seams" we see on the surface. If v is held constant, y is free to vary, so the grid curves running lengthwise along the surface correspond to keeping v constant.



23. (a) Instead of proceeding directly, we use Formula 3 of Theorem 10.2.3: $\mathbf{r}(t) = t\mathbf{R}(t) \Rightarrow$

$$\mathbf{v} = \mathbf{r}'(t) = \mathbf{R}(t) + t\mathbf{R}'(t) = \cos \omega t \mathbf{i} + \sin \omega t \mathbf{j} + t\mathbf{v}_d.$$

- (b) Using the same method as in part (a) and starting with $\mathbf{v} = \mathbf{R}(t) + t\mathbf{R}'(t)$, we have

$$\mathbf{a} = \mathbf{v}' = \mathbf{R}'(t) + \mathbf{R}'(t) + t\mathbf{R}''(t) = 2\mathbf{R}'(t) + t\mathbf{R}''(t) = 2\mathbf{v}_d + t\mathbf{a}_d.$$

- (c) Here we have $\mathbf{r}(t) = e^{-t} \cos \omega t \mathbf{i} + e^{-t} \sin \omega t \mathbf{j} = e^{-t} \mathbf{R}(t)$. So, as in parts (a) and (b),

$$\mathbf{v} = \mathbf{r}'(t) = e^{-t} \mathbf{R}'(t) - e^{-t} \mathbf{R}(t) = e^{-t} [\mathbf{R}'(t) - \mathbf{R}(t)] \Rightarrow$$

$$\begin{aligned} \mathbf{a} = \mathbf{v}' &= e^{-t} [\mathbf{R}''(t) - \mathbf{R}'(t)] - e^{-t} [\mathbf{R}'(t) - \mathbf{R}(t)] = e^{-t} [\mathbf{R}''(t) - 2\mathbf{R}'(t) + \mathbf{R}(t)] \\ &= e^{-t} \mathbf{a}_d - 2e^{-t} \mathbf{v}_d + e^{-t} \mathbf{R} \end{aligned}$$

Thus, the Coriolis acceleration (the sum of the "extra" terms not involving \mathbf{a}_d) is $-2e^{-t} \mathbf{v}_d + e^{-t} \mathbf{R}$.

24. By the Fundamental Theorem of Calculus, $\mathbf{r}'(t) = \langle \sin(\pi t^2/2), \cos(\pi t^2/2) \rangle$, $|\mathbf{r}'(t)| = 1$ and so $\mathbf{T}(t) = \mathbf{r}'(t)$.

Thus $\mathbf{T}'(t) = \pi t \langle \sin(\pi t^2/2), \cos(\pi t^2/2) \rangle$ and the curvature is $\kappa = |\mathbf{T}'(t)| = \sqrt{(\pi t^2)^2 (1)} = \pi |t|$.

$$25. (a) F(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ \sqrt{1-x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ -x + \sqrt{2} & \text{if } x \geq \frac{1}{\sqrt{2}} \end{cases} \Rightarrow F'(x) = \begin{cases} 0 & \text{if } x < 0 \\ -x/\sqrt{1-x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ -1 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases} \Rightarrow$$

$$F''(x) = \begin{cases} 0 & \text{if } x < 0 \\ -1/(1-x^2)^{3/2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ 0 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases}$$

$$\text{since } \frac{d}{dx} [-x(1-x^2)^{-1/2}] = -(1-x^2)^{-1/2} - x^2(1-x^2)^{-3/2} = -(1-x^2)^{-3/2}.$$

Now $\lim_{x \rightarrow 0^+} \sqrt{1-x^2} = 1 = F(0)$ and $\lim_{x \rightarrow (1/\sqrt{2})^-} \sqrt{1-x^2} = \frac{1}{\sqrt{2}} = F\left(\frac{1}{\sqrt{2}}\right)$, so F is continuous. Also, since

$$\lim_{x \rightarrow 0^+} F'(x) = 0 = \lim_{x \rightarrow 0^-} F'(x) \text{ and } \lim_{x \rightarrow (1/\sqrt{2})^-} F'(x) = -1 = \lim_{x \rightarrow (1/\sqrt{2})^+} F'(x), F' \text{ is continuous. But}$$

$$\lim_{x \rightarrow 0^+} F''(x) = -1 \neq 0 = \lim_{x \rightarrow 0^-} F''(x), \text{ so } F'' \text{ is not continuous at } x = 0. \text{ (The same is true at } x = \frac{1}{\sqrt{2}}.)$$

So F does not have continuous curvature.

- (b) Set $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. The continuity conditions on P are $P(0) = 0$, $P(1) = 1$, $P'(0) = 0$ and $P'(1) = 1$. Also the curvature must be continuous. For $x \leq 0$ and $x \geq 1$, $\kappa(x) = 0$; elsewhere

$$\kappa(x) = \frac{|P''(x)|}{(1 + [P'(x)]^2)^{3/2}}, \text{ so we need } P''(0) = 0 \text{ and } P''(1) = 0.$$

The conditions $P(0) = P'(0) = P''(0) = 0$ imply that $d = e = f = 0$.

The other conditions imply that $a + b + c = 1$, $5a + 4b + 3c = 1$, and $10a + 6b + 3c = 0$. From these, we find that $a = 3$, $b = -8$, and $c = 6$.

Therefore $P(x) = 3x^5 - 8x^4 + 6x^3$. Since there was no solution with $a = 0$, this could not have been done with a polynomial of degree 4.

