

**Math 21a Practice Final Exam**  
(Fall 2000)

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Name: \_\_\_\_\_

Section TF:

Allcock • Chen • Karigiannis • Knill • Liu • Rasmussen • Rogers • Taubes • Winter • Winters

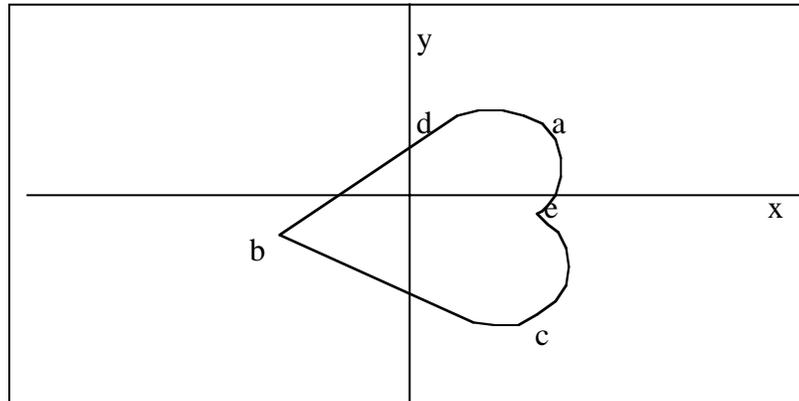
**Instructions:**

- Print your name in the line above and circle the name of your section TF.
  
- **THERE ARE THREE PARTS TO THIS EXAM:**
  - A. All students answer Part A questions.**
  - B. Only students in either the Regular or Physics sections answer Part B questions.**
  - C. Only students in the BioChem sections answer Part C questions.**
  
- Answers are to be written on the same page as the question. However, if more space is needed, use the back of the facing page. Use pages with unanswered questions (either those in Part C or in Part B, as the case may be) if additional space is needed.
- Please write neatly. Answers which are deemed illegible by the grader will not receive credit.
- No calculators, computers or other electronic aids are allowed; nor are you allowed to refer to any written notes or source material; nor are you allowed to communicate with other students.
- All problems will count the same amount.
- Neither unstaple nor remove pages from your exam booklet.

*In agreeing to take this exam, you are implicitly accepting Harvard University's honor code.*

## PART A

1.



2. a)  $\int_0^{1/\sqrt{2}} \left( \int_y^{(1-y^2)^{1/2}} \left( \int_z^{(1-x^2-y^2)^{1/2}} xyz dz \right) dx \right) dy.$

b)  $\int_0^1 \left( \int_0^{\pi/4} \left( \int_0^r r^2 z \cos \theta \sin \theta dz \right) d\theta \right) r dr.$

c)  $\int_0^1 \left( \int_0^{\pi/2} \left( \int_0^{\pi/4} \rho^3 \cos \phi \sin^2 \phi \cos \theta \sin \theta d\theta \right) \sin \phi d\phi \right) \rho^3 d\rho.$

3. a) (2, 2, 1) is one such vector. The vector (2, 7, 0) is another.

b)  $-7x + 2y + 10z = 0.$

c)  $L(x, y, z) = 6 + (-7x + 2y + 10z)/9.$

d)  $5/(9\sqrt{3}).$

4. 1/12.

5. The square of the norm of the gradient is  $x^2/32 + y^2/4 + 4z^2$ . This has its maximum on the boundary at  $(0, 0, \pm 1)$ .

6. One such equation is  $t \rightarrow (1 + 3t, -2t, t)$ .

7. 2/15

8. a) A; if  $\nabla u = 0$ , then  $u$  is constant and  $u_{xx}$  and  $u_{yy}$  are both zero.

b) S; for example  $u = y^2/2$  solves the equation as does  $u = x^2/2$ .

c) A; At a global maximum,  $u_{xx} + u_{yy} \leq 0$ .

d) N; the origin would then be a local maximum so  $u_{xx} + u_{yy} \leq 0$  there.

e) S; this is the case for  $(x^2 + y^2)/4$ , but not the case for  $(x^2 + y^2)/4 - 3000x$ .

## PART B

9. a) The flux through B is bigger. Here is why: First,  $\iint_A \mathbf{F} \cdot \mathbf{n} \, dS + \iint_B \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_V \operatorname{div}(\mathbf{F}) \, dV$  where  $\mathbf{n}$  points outwards from the region between A and B. Thus,  $\mathbf{n} \cdot (0, 0, 1) > 0$  on B and  $\mathbf{n} \cdot (0, 0, 1) < 0$  on A. Meanwhile  $\operatorname{div}(\mathbf{F}) = 2 + 2z > 0$  on V. Therefore, the A flux in question (computed with  $-\mathbf{n}$ ) must be less than the B flux computed with  $\mathbf{n}$ .
- b) The flux of  $\operatorname{curl}(\mathbf{F})$  is the same through both since they both have the same boundary (use Stokes' theorem).
- c) Since the disk where  $z = 0$  and  $x^2 + y^2 \leq 1$  has the same boundary as the surface under consideration, one can just as well compute the flux through this disk. The normal to this disk is  $(0, 0, 1)$  and  $\operatorname{curl}(\mathbf{F})$  where  $z = 0$  is  $(y, x, 0)$  and so the flux of  $\operatorname{curl}(\mathbf{F})$  is zero.

10. a) Parametrize the surface by  $(u, v)$  where  $0 \leq u \leq 1$  and  $0 \leq v \leq 2\pi$  according to the rule  $(u, v)$

$\rightarrow (u, u/2 \cos \theta, u/2 \sin \theta)$ . Then, the surface area is  $\int_0^{2\pi} \left( \int_0^1 \frac{\sqrt{5}}{4} u \, du \right) d\theta$ .

b)  $-\int_0^{2\pi} \left( \int_0^1 \frac{1}{4} u^2 \, du \right) d\theta$

11. a)  $\mathbf{v} = (z, -z, 0)$   
 b) Not possible since a unit radius disk,  $S$ , in this plane has normal  $(1, 0, 0)$  whose dot product with  $(1, 1, 0)$  is 1. Thus, the flux of the hypothetical vector field through  $S$  would equal  $\pi$  and so, by Stokes' theorem, its line integral over the boundary circle would not be zero.  
 c)  $\mathbf{v} = (2x, 0, 0)$ .  
 d) Not possible by the divergence theorem: The flux through any closed surface would equal twice the volume of the enclosed region.

12. a) By Green's theorem,  $\mathbf{v} = (0, x^3/3)$ .

b)  $\int_0^{2\pi} \frac{1}{3} \cos^4 t dt = \pi/4$ .

## PART C

For Problems 9 and 11, you can refer to the table that is provided at the back of the exam.

9. a)  $\mu = 56$  and  $\sigma = 16.8$ .

b)  $\sum_{m=62}^{80} \binom{80}{m} (.7)^m (.3)^{80-m}$ .

c)  $\int_{62}^{\infty} \frac{1}{\sqrt{2\pi}16.8} \exp(-(x - 56)^2 / 33.6) dx$

10. a)  $26/27$ .

b)  $1/54$ .

c)  $1/9$ .

11. a)  $(12.5)^k e^{-12.5}/k!$ .

b) Since  $\mu = 25$  and  $\sigma = 25$ , this is  $3/5$  standard deviations from the mean.

c) The probability of 40 deaths in one 4-year span is  $(25)^{40} e^{-25}/(40!)$ . The probability of 40 deaths in two consecutive four year spans is the square of this last number.

12. a) 10.3 percent.

b) Sensitivity =  $0.013/0.103 \sim 0.13$ .  $PV^+ = 0.013/0.1 = 0.13$ .

c) 0.1.