Answers to Math 21a (Fall 99) Review Problems:

1. a) $x - (2 - 2y + y^2)^{1/4} = 0$
   b) $(x^2 + 4)^{1/2} - 2y = 0$.
   c) $x^2/16 + y^2/9 = 1$.

2. a) $x = 1$. (This was changed.)
   b) $27x + 2y = -73$. (This was changed.)
   c) $x + 3y = -1$.

3. $e^2 + 1$.

4. a) $(t\cos(t^3), t\sin(t^3))$.
   b) $x\sin((x^2 + y^2)^{3/2}) - y\cos((x^2 + y^2)^{3/2}) = 0$.

5. a) $\mathbf{B} = 1/3\ \mathbf{A} + 1/3\ (2, 4, -5)$.
   b) $\mathbf{B} = 11/25\ \mathbf{A} + 1/25\ (92, 69, 25)$.
   c) $\mathbf{B} = -1/3\ \mathbf{A} + 1/3\ (11, 8, 7)$.

6. Only in Case c) are **v** and **w** perpendicular.

7. a) 2/3.
   b) $(13)^{1/2}/7$.

8. a) $2x + y - z = 2$.
   b) $x + y - z = 3$.
   c) $x - y = -1$.

9. 6/7.

10. $t \to (t, t, t)$ or $(-t,-t,-t)$ (This was changed.)

11. a) Yes. b) No. c) Yes. The answer is yes if there is a constant, non-zero vector which is orthogonal to **v** at each time t. Otherwise, the answer is no. For a), consider (-1, 0, 5) and for c), consider (0, 7, 1). No such vector exists for b) since in this case, $\mathbf{v}(0) = (0, 0, 1)$ and so such a vector would have to lie in the x-y plane. But then it couldn't be simultaneously orthogonal to $\mathbf{v}(\pi/2)$ and $\mathbf{v}(-\pi/2)$.

12. a) In the first case, $L = 20x + y - z - 9$. In the second, $L = 20x - y + 3z - 7$. (This was changed.)
    b) In the first case, $L = z + 1$. In the second, $L = 3y + z - 3$.

13. a) In the first case, the plane is where $z = 1$. In the second, it is where $x + z = 1$.
    b) In the first case, the plane is where $2x - y = 2$. In the second, it is where $x + y - z = 1$.

14. $L = 2x - 5y + z$.

15. a) 2. b) -1. c) -2.

16. $\nabla G = (-2, -4)$.

17. a) $\nabla f = (-\sin(x), 2y)$ so possible the critical points have the form $(n\pi, 0)$ with n an integer. The second derivative tests finds $(n\pi, 0)$ a local minimum when n is odd and a saddle when n is even.
    b) $\nabla f = (-\sin(x)\sin(y), \cos(x)\cos(y))$ so the critical points have the form $(n\pi, (m + 1/2)\pi)$ and $((n + 1/2)\pi, m\pi)$ where n and m are integers. In the first case, if both n and m are odd or both are even, it is a local maximum. If one is odd and the other not, then it is a local minimum. In the second case, it is a saddle for all n and m. (Use the 2^{nd} derivative test.)
    c) $\nabla f = (2x, 3y^2 - 3)$ so the critical points are $(0, \pm 1)$. The point (0, 1) is a local minimum and the point (0, -1) is a saddle. [This was changed.]

18. The origin is the only interior critical point and it is a saddle. The extreme points on the boundary occur at $(\pm 3/\sqrt{2}, \pm\sqrt{2})$. The points $(3/\sqrt{2}, \sqrt{2})$ and $(-3/\sqrt{2}, -\sqrt{2})$ are maxima and the others are minima.

19. The maxima occur at $(\pm 1, 0, 0)$ and the minima at $(0, \pm 1, 0)$.

20. The maximum is at (1, 0, 0).

21. Version 1: The closest points are at (1, 0, 0) and (-1, 0, 0).
    Version 2:The closest points are $(0, \pm 1, \pm 1)$ with distance $\sqrt{2}$.

22. The square of the length of **E** is $x^4 + 4y^2 + z^2 + z^2y^2 - 2zy + 1$ which is smallest at the origin.

23. $3(\pi/2 - 1)$.

24. $4/(3\pi)$. (Remember to divide the integral of x by the area.)

25. $\pi/8$.

26. $\sin(1)/2$.

27. $\pi/2$

28. $\pi \sin(1)$.

29. $4\pi (1 - \cos(1))/3$.

30. $\int_0^1 \left(\int_0^r \left(\int_0^\pi \left(r^3 \sin(\theta) \cos^2(\theta) \right) d\theta \right) dz \right) r\, dr = 1/9$.

31. $\int_0^1 \left(\int_0^{\pi/2} \left(\int_0^{\pi/2} \left(r^4 \sin(\theta) \cos(\theta) \sin^2(\phi) \cos^2(\phi) \right) d\theta \right) \sin(\phi)\, d\phi \right) r^2\, dr = 1/105$.

32. $\int_{-3/5}^{3/5} \left(\int_{-\sqrt{9/25-x^2}}^{\sqrt{9/25-x^2}} \left(\int_{\frac{4}{3}\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz \right) dy \right) dx = \int_0^{3/5} \left(\int_{\frac{4}{3}r}^{\sqrt{1-r^2}} \left(\int_0^{2\pi} d\theta \right) dz \right) r\, dr$

$= \int_0^1 \left(\int_0^{\mathrm{Arc\,tan}(3/4)} \left(\int_0^{2\pi} d\theta \right) \sin(\phi)\, d\phi \right) r^2\, dr = 2\pi/15$.

33. a) $\int_0^\infty \int_0^{2\pi} e^{-r^2}\, d\theta\, r\, dr = \pi$.

b) The square of this integral is the Cartesian coordinate version of the preceding integral, so the integral in question equals $\pi^{1/2}$.

34. $u(t, x) = e^t (1 + (x - 4t)^2)^{-1}$.

35. a) Advection. b) Advection. c) Diffusion.

36. $2^{1/2} \pi$

37. 16/7

38. $2/5\pi - 1/3$.

39. a) $\mathbf{X}(u, v) = (2 (1 - u^2/9 - v^2/25)^{1/2}, u, v)$ for values of (u, v) with $u^2/9 + v^2/25 \le 1$.
b) $\mathbf{X}(u, v) = (u, 3 (1 - u^2/4 - v^2/25)^{1/2}, v)$ for values of (u, v) with $u^2/4 + v^2/25 \le 1$.
c) $\mathbf{X}(u, v) = (1 - u^2 - v^4, u, v)$ for values of (u, v) with $u^2 + v^4 \le 1$.
d) $\mathbf{X}(u, v) = (u, v, 2 - u - v)$ for values of (u, v) in the first quadrant.

40\. a) $\int_{-5}^{5}\left(\int_{-3\sqrt{1-v^2/25}}^{3\sqrt{1-v^2/25}} \sqrt{1+(u^2/81+v^2/625)/(1-u^2/9-v^2/25)}\, du\right) dv.$

b) $\int_{-5}^{5}\left(\int_{-2\sqrt{1-v^2/25}}^{2\sqrt{1-v^2/25}} \sqrt{1+(u^2/16+v^2/625)/(1-u^2/4-v^2/25)}\, du\right) dv.$

c) $\int_{-1}^{1}\left(\int_{-\sqrt{1-v^4}}^{\sqrt{1-v^4}} \sqrt{1+4u^2+16v^6}\, du\right) dv.$

41\. 5/2. (The area of the surface is 2π and the integral of z over the surface is 5π.)

42\. sin(1).

43\. $20\pi/3$.

44\. (x, -y, 0).

45\. a) ($x^2yz/2$, 0, 0).
b) (2z, 3x, y).

46\. a) (0, x/π).
b) (x, 0, 0).

47\. a) No such vector field exists because (x, -2y, xy) has divergence -1 and the divergence of a curl is zero.
b) (xy cos(yz^2), 0, 0).

48\. 5x + 3y = 0.

49\. $\iint_S x\, g\, dS$. Let **E** = (g, 0, 0); here div(**E**) = f while **E•n** = x g on the surface of the volume, V, in question. Thus, the divergence theorem implies that $\iint_S x\, g\, dS = \iiint_V f\, dV$.

50\. a) 2π . This is a direct computation: Parameterize the circle by $t \to (\cos(t), \sin(t))$ and then path integral is just the integral of dt between 0 and 2π.
b) 2π. Use Green's theorem for a region with holes and note that the vector field in question has zero 'curl' in the sense that when written as (P, Q), then $Q_x - P_y = 0$.
c) 0. This is also a direct application of Green's theorem.