Answers to Math 21a Spring 2000 Review Problems:

1. a) $x - (2 - 2y + y^2)^{1/4} = 0$
   b) $(x^2 + 4)^{1/2} - 2y = 0$.
   c) $x^2/16 + y^2/9 = 1$.

2. a) $x = 1$.
   b) $27x + 4y = -73$.
   c) $x + 3y = -1$.

3. $e^2 + 1$.

4. a) $(t \cos t^3, t \sin t^3)$.
   b) $x \sin (x^2 + y^2)^{3/2} - y \cos (x^2 + y^2)^{3/2} = 0$.

5. a) $\mathbf{B} = 1/3\, \mathbf{A} + 1/3\, (2, 4, -5)$.
   b) $\mathbf{B} = 11/25\, \mathbf{A} + 1/25\, (92, 69, 25)$.
   c) $\mathbf{B} = -1/3\, \mathbf{A} + 1/3\, (11, 8, 7)$.

6. Only in Case c) are **v** and **w** perpendicular.

7. a) 2/3.
   b) $(13)^{1/2}/7$.

8. a) $2x + y - z = 2$.
   b) $x + y - z = 3$.
   c) $x - y = -1$.

9. 6/7.

10. $t \to (t, t, t)$ or $(-t,-t,-t)$

11. a) Yes b) No c) Yes. The answer is yes if there is a constant, non-zero vector which is orthogonal to **v** at each time *t*. Otherwise, the answer is no. For a), consider (-1, 0, 5) and for c), consider (0, 7, 1). No such vector exists for b) since in this case, $\mathbf{v}(0) = (0, 0, 1)$ and so such a vector would have to lie in the *xy*-plane. But then it couldn't be simultaneously orthogonal to $\mathbf{v}(\pi/2)$ and $\mathbf{v}(-\pi/2)$.

12. a) In the first case, $L = 20x + y - z - 9$.
    In the second, $L = 20x - y + 3z - 7$.
    b) In the first case, $L = z$.
    In the second, $L = 3y + z$.

13. a) In the first case, the plane is where $z = 1$.
    In the second, it is where $x + z = 1$.
    b) In the first case, the plane is where $2x - y = 2$.
    In the second, it is where $x + y - z = 1$.

14. $L = 2x - 5y + z$.

15. a) 2. b) -1. c) -2.

16. $\nabla G = (-2, -4)$.

17. a) $\nabla f = (-\sin x, 2y)$, so possible critical points have the form $(n\pi, 0)$ with n an integer. The second derivative tests finds $(n\pi, 0)$ a local minimum when n is odd and a saddle when n is even.
    b) $\nabla f = (-\sin x \sin y, \cos x \cos y)$, so the critical points have the form $(n\pi, (m + 1/2)\pi)$ and $((n + 1/2)\pi, m\pi)$ where n and m are integers. In the first case, if both n and m are odd or both are even, it is a local maximum. If one is odd and the other not, then it is a local minimum. In the second case, it is a saddle for all n and m.
    (Use the 2^{nd} derivative test.)
    c) $\nabla f = (2x, 3y^2 - 3)$, so the critical points are $(0, \pm 1)$. The point (0, 1) is a local minimum and the point (0, -1) is a saddle.

18. The origin is the only interior critical point and it is a saddle. The extreme points on the boundary occur at $(\pm 3/\sqrt{2}, \pm\sqrt{2})$. The points $(3/\sqrt{2}, \sqrt{2})$ and $(-3/\sqrt{2}, -\sqrt{2})$ are maxima and the others are minima.

19. The maxima occur at (±1, 0, 0) and the minima at (0, ±1, 0).

20. The maximum is at (1, 0, 0).

21. a: The closest points are at (1, 0, 0) and (-1, 0, 0) with distance 1.
b: The closest points are (2, 0, 0) and (-2, 0, 0) with distance 2.

22. The square of the length of **E** is $x^4 + 4y^2 + z^2 + z^2y^2 - 2zy + 1$ which is smallest at the origin.

23. 3(π/2 - 1).

24. 4/(3π). (Remember to divide the integral of x by the area.)

25. π/8.

26. (sin 1)/2.

27. π/2

28. π (sin 1)

29. 4π(1 - (cos 1))/3.

30. $\int_0^1 \left(\int_0^r \left(\int_0^\pi \left(r^3 \sin\theta \cos^2\theta\right) d\theta\right) dz\right) r\, dr = 1/9.$

31. In spherical coordinates, we have $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^6 \sin\theta\cos\theta\sin^3\phi\cos^2\phi\, d\rho\, d\phi\, d\theta = \frac{1}{56}$

32. $\int_{-4/5}^{4/5}\left(\int_{-\sqrt{16/25-x^2}}^{\sqrt{16/25-x^2}}\left(\int_{\frac{3}{4}\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz\right)dy\right)dx = \int_0^{4/5}\left(\int_{\frac{3}{4}r}^{\sqrt{1-r^2}}\left(\int_0^{2\pi} d\theta\right) dz\right) r\, dr$

$= \int_0^1\left(\int_0^{\arctan(4/3)}\left(\int_0^{2\pi} d\theta\right)\sin\phi\, d\phi\right)\rho^2\, d\rho = 4\pi/15.$ (this was corrected May 19)

33. a) $\int_0^\infty \int_0^{2\pi} e^{-r^2}\, d\theta\, r\, dr = \pi.$

b) The square of this integral is the Cartesian coordinate version of the preceding integral, so the integral in question equals $\pi^{1/2}$.

34. 0 (this was corrected)

35. $-\frac{2}{\pi} - \frac{2}{7}$ (this was corrected).

36. 0 (this was corrected)

37. a) $\mathbf{X}(u, v) = (2\,(1 - u^2/9 - v^2/25)^{1/2}, u, v)$ for values of (u, v) with $u^2/9 + v^2/25 \le 1$.

b) $\mathbf{X}(u, v) = (u, 3\,(1 - u^2/4 - v^2/25)^{1/2}, v)$ for values of (u, v) with $u^2/4 + v^2/25 \le 1$.

c) $\mathbf{X}(u, v) = (1 - u^2 - v^4, u, v)$ for values of (u, v) with $u^2 + v^4 \le 1$.

d) $\mathbf{X}(u, v) = (u, v, 2 - u - v)$ for values of (u, v) in the first quadrant.

38. a) $\int_{-5}^{5}\left(\int_{-3\sqrt{1-v^2/25}}^{3\sqrt{1-v^2/25}} \sqrt{1+4(u^2/81+v^2/625)/(1-u^2/9-v^2/25)}\,du\right)dv$. (This was corrected.)

b) $\int_{-5}^{5}\left(\int_{-2\sqrt{1-v^2/25}}^{2\sqrt{1-v^2/25}} \sqrt{1+9(u^2/16+v^2/625)/(1-u^2/4-v^2/25)}\,du\right)dv$. (This was corrected.)

c) $\int_{-1}^{1}\left(\int_{-\sqrt{1-v^4}}^{\sqrt{1-v^4}} \sqrt{1+4u^2+16v^6}\,du\right)dv$.

39. This average is calculated over the disk D in the xy-plane with $x^2 + y^2 \le 1$. For each point (x, y) in this disk, the height is given by $z = 5\sqrt{1-x^2-y^2} = 5\sqrt{1-r^2}$. The area of the base disk is π. The average is given by $\dfrac{\int_D z(x,y)\,dA}{area(\mathrm{D})} = \dfrac{10}{3}$.

40. sin 1

41. $\iint_{S=\partial B} \mathbf{F}\bullet\mathbf{n}\,dS = \iiint_B div\,\mathbf{F}\,dV$

$= \iiint_B 1\,dV = \dfrac{500\pi}{3}$

(this problem was corrected)

42. $(x, -y, 0)$

43. a) $(x^2yz/2, 0, 0)$.

b) $(2z, 3x, y)$.

44. a) $(0, x/\pi)$

b) $(x, 0, 0)$

45. a) No such vector field exists because $(x, -2y, xy)$ has divergence -1 and the divergence of a curl is zero.

b) $(xy \cos yz^2, 0, 0)$.

46. $5x + 3y = 0$

47. $\iint_S xg\,dS$. Let $\mathbf{E} = (g, 0, 0)$; here div $\mathbf{E} = f$ while $\mathbf{E}\bullet\mathbf{n} = xg$ on the surface of the volume, V, in question. Thus, the divergence theorem implies that $\iint_S xg\,dS = \iiint_V f\,dV$.

48. a) 2π. This is a direct computation: Parameterize the circle by $t \to (\cos t, \sin t)$ and then the path integral is just the integral of dt between 0 and 2π.

b) 2π. Use Green's theorem for a region with holes and note that the vector field in question has zero 'curl' in the sense that when written as (P, Q), then $Q_x - P_y = 0$.

c) 0. This is also a direct application of Green's theorem.